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Santa Barbara

An Experiment in the Aesthetic Value of Sonified Mathematical Objects

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy
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by

Casey Patrick Mongoven

Committee in charge:
Professor Curtis Roads, Chair
Professor Clarence Barlow
Professor Tobias Höllerer

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The dissertation of Casey Patrick Mongoven is approved.

Tobias Höllerer

Clarence Barlow

Curtis Roads, Committee Chair

August 2013

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VITA OF CASEY PATRICK MONGOVEN
September 2013

EDUCATION

Bachelor of Music in Classical Composition, New England Conservatory, May 2001 (with academic honors)

Ergänzungsstudium in Electroacoustic Composition, Hochschule für Musik Franz Liszt Weimar, January 2009

PROFESSIONAL EMPLOYMENT

2010–2013: Teaching Assistant, Department of Germanic, Slavic, and Semitic Studies, University of California, Santa Barbara

PUBLICATIONS

Andreas Werckmeister's *Cribrum musicum* (1700) and *Harmonologia musica* (1702). New York: Pendragon Press, 2013.

Sonification of Multiple Fibonacci-Related Sequences. *Annales Mathematicae et Informaticae* 41: 175–192, 2013.

Johann David Heinichen's *Gründliche Anweisung* (1711) (with Benedikt Brilmayer). New York: Pendragon Press, 2012.

Musical Composition with Zeckendorf Representations (with Ron Knott). *Aportaciones Matemáticas Investigación* 20: 199–216, 2011.

A Style of Music Characterized by Fibonacci [Numbers] and the Golden Ratio. *Congressus Numerantium* 201: 127–138, 2010.

CONCERTS/INSTALLATIONS

MAT End of Year Show: Shadows in Space (Goleta, CA)
Stolarsky Arrays no. 1, *Rep Sequences no. 19* May 2013

Fifteenth International Conference on Fibonacci Numbers and Their Applications (Eger, Hungary)
Five Pieces June 2012

MAT End of Year Show: Bits and Pieces (Goleta, CA)
Three Pieces May 2012

CREATE Concert: Critical Point (Goleta, CA)
Zeckendorf Representations no. 19 October 2011

MAT End of Year Show: Questionable Utility (Goleta, CA)
Zeckendorf Representations no. 18 June 2011

Fourteenth International Conference on Fibonacci Numbers and Their Applications (Morelia, Mexico)
Zeckendorf Representations nos. 17, 18 and 19 July 2010

CREATE Concert: Duck-Billed Platypus (Goleta, CA)
G(1,2) and G(1,1) Rep Sequences no. 1 February 2010

Diplomkonzert Neue Musik (Weimar, Germany)
(multiple works) January 2009

Installation in the Franz Liszt Museum (Weimar, Germany)
Fibonacci and Lucas in White 2008–2009

Thirteenth International Conference on Fibonacci Numbers and Their Applications (Patras, Greece)
Seven Pieces from Collection VIII July 2008

SeaM Konzert: Klangbilder (Sondershausen, Germany)
13 Stücke aus der Collection VI July 2008

SeaM Konzert: Elektroakustische Musik, Fürstensaal (Weimar, Germany)
Zeckendorf Representations no. 16, *Drei Stücke aus der Collection V* July 2007

KONZERTE: DEGEM WebRadio @ ZKM (Karlsruhe, Germany)
G(1,2) Rep Sequence no. 18 June 2007

Tuesday Night New Music (Boston, MA)
Chaos and Pisano's Rabbit Problem April 2001

Tuesday Night New Music (Boston, MA)
Study in Fibonacci Pitch Class Sequences no. 5 October 2000

Tuesday Night New Music (Boston, MA)
3 Pieces for Violin March 2000

Tuesday Night New Music (Boston, MA)
Duet for Disklaviers March 1999

AWARDS

2012–2013 Teaching Assistant of the Year, Department of Germanic, Slavic, and Semitic Studies, University of California, Santa Barbara, 2013

2011–2012 Teaching Assistant of the Year, Department of Germanic, Slavic, and Semitic Studies, University of California, Santa Barbara, 2012

2010–2011 Teaching Assistant of the Year, Department of Germanic, Slavic, and Semitic Studies, University of California, Santa Barbara, 2011

Nomination into Pi Kappa Lambda, National Music Honors Society, 2001

FIELDS OF STUDY

Major Field: Media Arts and Technology

Studies in Signal Processing with Curtis Roads

Studies in Algorithmic Composition with Clarence Barlow

Studies in Experimental Design and Statistics with Tobias Höllerer

ABSTRACT

An Experiment in the Aesthetic Value of Sonified Mathematical Objects

by

Casey Patrick Mongoven

A central problem in the field of aesthetics is the determination of what we find to be beautiful and why. Since the work of Adolf Zeising and Gustav Theodor Fechner in the mid-nineteenth century, there has been great research interest in determining the potential aesthetic merits of the golden ratio and the Fibonacci sequence. A review of the literature shows that a large body of research exists that studies their aesthetic value in the visual domain. No research has yet been carried out, however, in an attempt to determine their potential merits in the audio domain. This has been due in part to a lack of suitable example stimuli for scientific testing. This dissertation takes the first step in bridging this research gap, guided by the question: Do sonifications of mathematical objects closely related to FS and GR hold some special aesthetic significance or advantage over less closely related sonified mathematical objects? In the most comprehensive survey of its kind, works in which composers are known to have utilized the golden ratio and Fibonacci numbers are shown to be unsuitable for our research purposes, as they were never intended for such research. Methods are then outlined for creating sonifications that can function as the necessary simple example stimuli in experiments. After a presentation of various sonifications related to the golden ratio and Fibonacci sequence, the methodology and results of a novel experiment involving 170 participants are then detailed. In this experiment, subjects were asked to compare the relative aesthetic value of pairs of sonifications of mathematical objects. In each pair, one of the mathematical objects was closely related to the Fibonacci sequence whereas the other was not. The evidence gathered suggests that the utilization of the golden ratio and the Fibonacci sequence could have potentially negative effects on the aesthetic value of sonifications.

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1 The Generalized Fibonacci Sequence and the Golden Ratio

In this section, notation is introduced and enough preliminary mathematical background is given in order to familiarize the reader with the concepts discussed in the two sections that follow. A discussion of more advanced Fibonacci-related mathematics is reserved until later, when it will be more useful for analysis and composition.

1.1 The Golden Ratio

We shall begin with a geometric definition of the golden ratio (henceforth GR). We define the division of a line segment AC to be by GR when we divide it at the point B such that the resulting line segments have the property BC:AB = AB:AC.

Figure 1: Line segment divided at GR.



GR is thus defined to be a pair of inversely related values: $\varphi = \frac{\sqrt{5}-1}{2}$ and $\Phi = \frac{\sqrt{5}+1}{2}$. The first known mention of GR was by Euclid in his *Elements* (c. 300 BCE). GR is also variously referred to as the golden section, the golden mean, and the divine proportion. Roger Herz-Fischler, author of *A Mathematical History of the Golden Number*, prefers the word *golden number* instead of *golden ratio*, as the latter indicates that multiple values are involved [1].[1] In this dissertation, GR will be used to refer to the two values of φ and Φ described above.

1.2 The Generalized Fibonacci Sequence

In this dissertation, wedge brackets <> will be utilized to denote a sequence and curly brackets {} to denote a set. In addition to the standard Fibonacci sequence $F_n = < 0, 1, 1, 2, 3, 5, 8, 13, \cdots >$ (A000045 in [2]), which we will also call FS, many composers have used generalized Fibonacci sequences as well. We define a generalized Fibonacci sequence (henceforth GFS) G_n to begin with two integers $G_0 = a$ and $G_1 = b$. Values for $0 > n > 1$ are given by the Fibonacci rule $G_n = G_{n-1} + G_{n-2}$. One can see the relation of FS to GFS in the following table:

Figure 2: GFS and its relation to FS.

n	0	1	2	3	4	5	6	7	...
G_n	a	b	$a+b$	$a+2b$	$2a+3b$	$3a+5b$	$5a+8b$	$8a+13b$	...

[1] GR is also sometimes called the *golden section* or the *golden mean*.

Just as in F_n, where $G_0 = 0$ and $G_1 = 1$, the ratio of two successive terms of any G_n where $G_0 \neq 0$ and $G_1 \neq 0$ converges to GR as n approaches infinity [3]:

$$\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \Phi$$

One G_n that has been the subject of significant mathematical research is the so-called Lucas sequence, where $G_0 = 2$ and $G_1 = 1$, named after mathematician Édouard Lucas. We will refer to this sequence $< 2, 1, 3, 4, 7, 11, 18, \cdots >$ (A000032 in [2]) as L_n, or LS.

We will also see composers make use of the inverse G_n sequence, in which $G_n = G_{n-2} - G_{n-1}$.

F_n was known in India at least several centuries prior to Leonardo Pisano's (aka Fibonacci) famous rabbit problem in 1202. A method for its formation was given by Virahāṅka (c. 600–800 CE) [4].

2 Who I Am and Why I Am Doing This

I first learned about FS and GR at Interlochen Arts Camp in 1996, in a class on 20th century music taught by John Boyle Jr. I was fascinated by the fact that one of my favorite composers – Béla Bartók – had used mathematics in the composition of his *Music for String Instruments, Percussion and Celesta* (1936). The following year, upon entering New England Conservatory, I began composing works in which FS and GR lay at the core of the compositional process. Due to the geometric properties to be described in the previous section, I saw GR to be the ratio with the greatest theoretical potential to lend the coherence I sought to the formal structure of my compositions. Using the closely related FS in addition would in turn bring the level of coherence in my compositions to an even higher level, and provide for greater variety at the same time. As time went on and my catalog of compositions grew, I began to see FS and GR as a means of providing coherence in my oeuvre as well. My composition teacher at that time, Alan Fletcher, was very supportive of my ideas, and since that time, all of my compositions have integrated FS and GR in some form or fashion.

My first compositions were relatively traditional in nature, and utilized FS and GR at more of a surface level, much like the works of the composers we will see in section 4.2 of this dissertation. As time went on, I began to become interested in taking a more radical approach that would bring the integration of Fibonacci-related mathematical objects into musical structure to a level not yet achieved. In 2002, I composed my first sonifications, and a new and unique style was forged,[2] described in sections 5 and 6. At the time, I would not have used the word *sonification* to describe how I was composing, as I had never heard that word before. I considered the manner in which I was composing much more to be an extension of the principles of serial composition.

Upon contemplating a research question for this dissertation, I reflected upon the fact that I had always been interested by claims that GR held particular aesthetic value in the visual domain. I was aware of the large body of research devoted to its study in aesthetics. Given the research-oriented nature of my compositional output, it came completely naturally to regard a study of FS and GR's aesthetic value in the audio domain as an extension of this work. I also had a deep personal conviction that I was the right person to carry out such a study, as I had more experience than anyone else in sonification of Fibonacci-related mathematical objects, and because of my great respect for the scientific method and what it has accomplished for mankind. As such, I went to work on "composing" the experiment described in section 7 of this dissertation – lending a new meaning to the word *experimentation* in an artistic context. This experiment represents the first time a composer or – as far as I have been able to determine – an artist has carried out a scientific experiment on the aesthetic value of his own work, and the first experiment devoted to the study of the aesthetic value of FS and GR in the audio domain.

[2]For a digression into the definition of the word style, see Appendix A.

3 The Origins of the Research Problem

As indicated in the previous section, the mathematical roots of the research described in this dissertation stretch far back into history and span diverse cultures. The scientific study of the aesthetic value of FS and GR, however, has its origins in the work of two German intellectuals: Adolf Zeising (1810–1876) and Gustav Theodor Fechner (1801–1887). This section briefly summarizes their work as well as some of the research that followed, placing the original research carried out in this dissertation in a historical context. For those interested in a more thorough understanding of Zeising's and Fechner's work, Appendix B contains a significant number of translated excerpts of the most pertinent works of these authors on the topic of aesthetics and FS and GR.

3.1 Adolf Zeising

Adolf Zeising's name is closely associated with the concept of *golden numberism*, which refers to the set of claims regarding the purported use of GR in man-made objects or its purported manifestation in nature [5]. Within a three-year period, Zeising wrote three works that outlined his views on aesthetics in general and the role of GR in nature and art [6, 7, 8]: *Neue Lehre von den Proportionen des menschlichen Körpers* [New Doctrine on the Proportions of the Human Body] (1854), *Aesthetische Forschungen* [Aesthetic Research Studies] (1855), and *Das Normalverhältniss der chemischen und morphologischen Proportionen* [The Standard Ratio of Chemical and Morphological Proportions] (1856). The following is a summary of Zeising's main points in these writings:

- The concept of proportionality is intimately connected with the concept of beauty.
- GR is the basis of a previously unknown fundamental morphological law pervading all of nature and art.
- Among other things, GR underlies the proportionality of the human body as well as that of some animals, the leaf arrangement on plants, the form of various crystals, the planetary system, the proportions of architectural and sculptural works of art recognized as being the most beautiful, and the most aesthetically pleasing chords in musical harmony.
- Although GR is a standard morphological proportion, it does not occur exactly in real manifestations, rather there is always a deviation to some greater or lesser degree. This deviation allows for the inner richness of the idea to come to light in multifarious manifestations.
- GR is not the only ratio of aesthetic value, but represents an aesthetic ideal towards which all remaining ratios strive.

Despite Zeising's strong assertions about GR, he provided little or no scientific evidence for his claims. His writings soon became the subject of a vivid discussion in the fields of natural sciences, philosophy and art, and this lack of evidence provided Fechner with the opportunity to undertake research on the topic.

3.2 Gustav Theodor Fechner

Fechner's pioneering work in the field or experimental aesthetics began with several experiments designed to determine the aesthetic significance of GR in simple geometric objects. Fechner was skeptical of Zeising's assertion that GR held any aesthetic significance whatsoever. Three important writings of Fechner's on the subject are [9, 10, 11]: *Ueber die Frage des goldnenen Schnitts* [On the Golden Ratio Problem] (1865), *Zur experimentalen Aesthetik* [On Experimental Aesthetics] (1871), and *Vorschule der Aesthetik* [Introduction to Aesthetics] (1876). A summary of the main points of these writings pertaining to the research carried out in this dissertation are as follows:

- The only scientific approach to aesthetics – referred to by some as the study of beauty – is one in which all assertions are supported by facts learned through experiments. Such an approach begins with simple lessons learned about what pleases and what displeases. From there, all concepts and laws are supported by the information gathered, rather than beginning with general ideas about beauty and its place in the system of general concepts such as what is true and good. The former (experimental approach) is called a *bottom-up approach*, the latter *top-down*.

- Two types of aesthetic impressions need to be differentiated: direct and associative. *Direct impressions* are those that forms, colors, and sounds as well as their interrelationship make independent from their meaning. *Associative impressions* are those that such constituents make as the carrier of a certain significance, in the service of certain purposes and ideas. When both impressions amalgamate into a unified impression, the former is the *direct* and the latter the *associative factor* of the impression. *Associative influence* is when the latter comes into play with the former.

- In music, the direct factor of the impression plays the main role, whereas the associative factor only plays a supporting role. In visual art, it is vice versa.

- The aesthetic impression of a form can sustain modification through constitution or combination with other equal or different forms (e.g. the form of a window viewed independently vs. a window viewed as part of a larger structure such as a building). The study of this phenomenon, called *combinatorial influence*, is a fundamental task of the field of experimental aesthetics.

- Complicated examples, such as Raphael's *Sistine Madonna*, which Zeising considered to be one of the most outstanding examples of the utilization of GR,

are not suited to the study the aesthetic value of GR. Rather, one must begin with the simplest geometric objects. This is because the more perfectly isolated a form becomes, the more combinatorial and associative influence recedes.

- The exactness of the results in aesthetics cannot be taken as far as in the fields of astronomy and physics. With experimental methods, however, one can come as close as is possible.

- One cannot directly measure the pleasantness or unpleasantness of a ratio. This would require a unit measurement of like or dislike, which does not yet exist. One can, however, count the number of persons who prefer one ratio over another. The ratio that gathers the most votes is then that which is most pleasant for the class of people that participated in the experiment. Measurements gathered according to such a method can be introduced into calculation just as well as with any other scientific method of measurement.

- Three methods can be used to determine the aesthetic value of a ratio: the *method of choice*, the *method of creation*, and the *method of utilization*. In the method of choice, one presents subjects with various objects of different shapes simultaneously (e.g. rectangles varying in their aspect ratios) and has the subjects choose those that they find most aesthetically appealing. In the method of creation, one has the subjects create the shape that for them is the most aesthetically appealing (e.g. one has them create a rectangle with an aspect ratio they find most pleasant). In the method of utilization, one collects as many objects as possible of a certain form (e.g. rectangular objects) where the purpose or significance of the object does not have a significant effect on its form, and then one measures them in an attempt to determine the most pleasant form.

- The measure of aesthetics not only has an objective significance, but also a subjective one, in that one can judge and determine the less good or correct taste according to a greater or lesser deviation from the standard ratio determined in an experiment.

- According to experiments carried out, a rectangle formed according to GR is the most aesthetically pleasing. Those closest to an aspect ratio of 1:1 and those whose aspect ratio tends towards 1:∞ are the least aesthetically pleasing.

- Experiments designed to determine the most aesthetically pleasing ratio for other forms (e.g. the dimensions of a cross, or the division of a simple line), however, did not show GR to be particularly advantageous. Zeising's assertion about the aesthetic significance of GR is thus overstated, although it is nonetheless to be considered the first true discovery in the field of aesthetics.

- It is possible to prove through experiments that symmetry has a much more decisive role in creating aesthetically pleasing forms than GR.[3]

[3]The word *symmetry* is used here in reference to reflective symmetry (e.g. dividing a line into

3.2.1 Fechner's Experiment With Rectangles

Fechner's first experiment in aesthetics was designed to determine the aesthetic value of GR as an aspect ratio of simple rectangles. The experiment utilized the method of choice described above and involved 347 participants, counting the number of preferences for a certain aspect ratio as well as rejections (those that the participants could clearly exclude from being most aesthetically pleasing). Fechner presented the subjects simultaneously with ten rectangles made of white card stock with varying aspect ratios, but with exactly the same surface area. If a subject settled on one specific ratio for preference or rejection, then a 1 was entered into the table. If the subject fluctuated between two or three ratios, then Fechner entered $\frac{1}{2}$ or $\frac{1}{3}$, respectively, into the columns of the table pertaining to those ratios. The table that follows is a translation of the original.[4] Fechner used Fibonacci numbers to approximate GR: $F_9/F_8 = 34/21 \approx 1.61905$. The symbol $\odot$ was used to identify the rectangle that approximated GR:

Figure 3: Fechner's experiment with rectangles from *Vorschule der Aesthetik*.

Table from the Experiment with Ten Rectangles

(A aspect ratio, Z number of preferences, z number of rejections, m male, f female.)

A	Z		z		percent Z	
	m	f	m	f	m	f
$\frac{1}{1}$ □	6.25	4.0	36.67	31.5	2.74	3.36
$\frac{6}{5}$	0.5	0.33	28.8	19.5	0.22	0.27
$\frac{5}{4}$	7.0	0.0	14.5	8.5	3.07	0.00
$\frac{4}{3}$	4.5	4.0	5.0	1.0	1.97	3.36
$\frac{29}{20}$	13.33	13.5	2.0	1.0	5.85	11.35
$\frac{3}{2}$	50.91	20.5	1.0	0.0	22.33	17.22
$\frac{34}{21}$ $\odot$	78.66	42.65	0.0	0.0	34.50	35.83
$\frac{23}{13}$	49.33	20.21	1.0	1.0	21.64	16.99
$\frac{2}{1}$	14.25	11.83	3.83	2.25	6.25	9.94
$\frac{5}{2}$	3.25	2.0	57.21	30.25	1.43	1.68
sum	228	119	150	95	100.00	100.00

In this particular study, GR was found to be the most aesthetically pleasing aspect

two parts of exactly equal size, or a face in which one side mirrors the other), and not in the more general mathematical or geometric sense.

[4]All translations in this dissertation are by the author.

ratio for a simple rectangle. None of the 347 subjects rejected GR and approximately 35% preferred it. The next most aesthetically pleasing aspect ratios were found to be those that most closely clustered around GR.

A study according to the third method, the method of utilization, in which Fechner measured various objects occurring in commerce (business cards, books, playing cards, chocolate bars, etc.), confirmed to a great extent what Fechner's experiment according to the first method had shown.

3.3 Additional Research on GR in the Visual Domain

Following Fechner's pioneering work in the field of experimental aesthetics, a number of similar studies were carried out by other researchers, many of whom had similar results to those of Fechner. A significant number, however, had results that contradicted Fechner's. This section briefly summarizes some of these researchers' contributions and results.

3.3.1 Lightner Witmer

In 1893, American psychologist Lightner Witmer (1867–1956) published *Zur experimentellen Aesthetik einfacher räumlicher Formverhältnisse* [On the Experimental Aesthetics of Simple Formal Proportions in Space]. In it, in addition to his own original research into the aesthetic value of GR, Witmer also describes the results of several research studies carried out by Fechner that had not been published. Witmer had gained access to Fechner's protocols and manuscripts while a student at the Universität Leipzig, where Fechner had served as Professor from 1834 until his death. The main points pertaining to the studies carried out by Fechner that were not published during his lifetime are as follows [12]:

- Fechner carried out an experiment with 76 subjects designed to determine the dimensions of the most aesthetically pleasing ellipse using the method of choice and, more specifically, the comparison of pairs using seven different ellipses drawn on a single piece of white card stock. The ratios of the major to minor axes ranged from 11.7:6.2 mm to 8.5:8.5 mm (from 1:1.887 to 1:1). The subjects were grouped into the following categories: older men (20 subjects), younger men (16 subjects), older women (20 subjects), younger women (20 subjects). GR was found to be the most aesthetically pleasing ratio for the major and minor axes of an ellipse, followed by a ratio of 2:3. No major differences in ascertainment were found among the categories of age.

- In a second experiment, two series of nine ellipses cut out of black and white card stock, respectively, were made in which the major axis remained unchanged (except for very minimal deviations less than two millimeters). The major to minor ratio of the axes ranged from 96:96 mm to 95:37.9 mm (from 1:1 to 2.507:1). A third series of nine ellipses was made of white card stock in which the minor axis remained unchanged. The minor to major ratio of these axes ranged from 60:60 mm to 60.8:149.4 mm (from 1:1 to 1:2.457). In this experiment,

the ellipses were presented horizontally, vertically and in random positions. 305 individual ascertainments were gathered. The ratio of 2:3 was found to be most aesthetically pleasant with 42.4% of all preferences, followed by GR with a mere 16.7%.

- Fechner often found ellipses on congratulatory cards and elegant letterheads. He measured 112 such ellipses according to the method of utilization and found, for the most part, that the ratios of their axes clustered around GR.

- Fechner carried out another experiment – the so-called *i*-experiments – on 69 subjects using 12 vertical lines 32 mm in length in which he placed a dot 1 mm in diameter directly above the line at varying heights. The ratio of line length to point height above the top of the line of 2:1 (32:16 mm) was found to be most pleasant followed by 2.48:1 (32:12.9 mm), both of which deviate significantly from GR. Similar results were obtained in an experiment utilizing the method of creation.

The following is a synopsis of Witmer's original research detailed in the mentioned writing:

- A new method of choice can be utilized in which, instead of a limited number of figures for comparison, a complete series with all gradations is used. While it would be impossible to include all gradations from 1:1 to 1:∞, experiments have shown that for most figures, zero votes of preference are found for values greater than 1:5. In addition, the phenomenon of just noticeable differences makes it unnecessary to have infinite gradations, so that one can consider a fixed number of figures (where the differences between the ratios in figures are no more than 5%) to represent a complete series for comparison.

- This new method has an advantage over Fechner's method because Fechner could have left out the most aesthetically pleasing figure. Fechner assumes in a series of figures $a, b, c, d, \ldots$ ordered according to size, that if the most pleasant ratio lies between c and d, that the preferences would be distributed between the neighboring figures c and d. This assumes, however, that the pleasantness of a group of figures is a continuous function of the size of the ratio, which cannot be proved. Depending on the figure, a could in fact be more aesthetically pleasing than c or d.

- Associative influence can be compensated for by presenting figures to subjects according to this new method, especially when they are presented with all gradations simultaneously ordered according to their size. In such a way, the subjects focus more on the proportions themselves than if one were to present them individually or two at a time. Subjects tended to be frustrated by a simultaneous presentation of a random array of series of figures with no order. This new method of presentation can even have the effect of intensifying and concentrating the aesthetic feelings associated with the figures.

- Two separate experiments, each involving two participants, were carried out in order to investigate the differences between pairwise comparison of rectangles vs. a comparison according to the new method of choice described above using simultaneous presentation of a complete series ordered according to the size of the ratio.

- In a first experiment, a single subject who had already taken part in other experiments using the new method (with simultaneous presentation) made 400 pairwise comparisons of 16 rectangles ranging from 50:23 mm to 50:38 mm (with constant width of 50 mm and gradations of height of 1 mm), the ratio of 2:1 (50:25 mm) was found to be most pleasant. Another subject who had not taken part in such experiments tended towards the rectangle that corresponded almost exactly to GR.

- As a follow-up to this study of pairwise comparisons, six studies were set up for each of the same two subjects over the course of six months in order to compensate for the influence of memory. These six studies were not pairwise but presented all of the figures simultaneously according to the new method. For each subject, the results were very similar to those obtained in the pairwise comparison test.

3.3.2 Charles Lalo

In 1908, the French philosopher Charles Lalo published a book entitled *L'esthétique expérimentale contemporaine* in which he detailed the results of several experiments [13]. The following is a translation of the table showing the results obtained in an experiment using the method of choice that more or less duplicated Fechner's original experiment:

Figure 4: Lalo's experiment with rectangles from *L'esthétique expérimentale contemporaine* (method of choice).

New Experiments on the Choice of Rectangles

RATIOS of the RECTANGLES' SIDES	1/1	6/5	5/4	4/3	29/20	3/2	34/21	23/13	2/1	5/2	TOTAL
Preferences	11.66	1	1.33	9.50	5.58	11	30.25	6.33	8	15.33	105
Indifference	6	4	3.66	7	2.33	7.50	4.75	1.50	3	1.25	41
Rejections	18	13.25	7.25	7.25	2	0.50	0	0.50	10	21.25	80
Attention received	**35.66**	18.25	12.24	**23.75**	9.91	19.10	**35.00**	8.33	21	**37.83**	226

Lalo notes the following regarding this experiment:

- Fechner's results are confirmed for the most part.

- GR gained the majority of preferences and – as in Fechner's study – was never rejected, but subjects also often declared indifference towards it.
- The extremes are more often chosen and excluded than in Fechner's experiment.
- Subjects gave varying reasons for their choices, such as the following: proportional/mathematical regularity, nobleness, austerity, mass, banality, heaviness, elegance, distinction.

Lalo also carried out an experiment using the method of creation, where 29 subjects were given a piece of white paper with one side fixed at 5 cm and told to construct the most aesthetically pleasing rectangle where the other side is between 5 to 10 cm. The following is the table showing the results of this experiment:

Figure 5: Lalo's experiment with rectangles from *L'esthétique expérimentale contemporaine* (method of creation).

Experiments on the Construction of Rectangles (fixed side: 5cm)

Dimensions of the variable side (in *cm*)	5 *cm*	5–5.5	5.5–6	6–6.5	6.5–7	7–7.5	7.5–8	8–8.5	8.5–9	9–9.5	9.5–10
Distribution of the constructed rectangles	0	1	0	2	3	7	3	5	2	6	0

Regarding this study, Lalo points out the following.

- The arithmetical mean of the results is 5:7.7, whereas GR is closer to 5:8.
- The results cluster to some degree around GR, and the extremes are rarely seen.

Lalo carried out informal experiments using the method of utilization, measuring various objects. He makes the following points pertaining to these experiments:

- This third method gives similar results to the two others, but is more problematic because of influences such as purpose and significance. The results therefore vary more for all sorts of objects.
- Although the method of utilization is the least reliable for judging aesthetic value, sociologically it is perhaps the most interesting because of what it might say about different cultures.
- The third method is also useful since it is a statistical method that has nothing to do with psychology or experiments.

3.3.3 Edward L. Thorndike

In 1917, American psychologist Edward Lee Thorndike (1874–1949) published the findings of several experiments in a paper called *Individual Differences in Judgments of the Beauty of Simple Forms* [14]. In it, he details the results of several experiments similar to Fechner's. The main points are as follows:

- In an experiment with 12 rectangles presented in order of decreasing aspect ratio simultaneously and horizontally to 200 individuals, an aspect ratio of 1.83:1 was found to be the most aesthetically appealing.
- In a similar experiment with 12 isosceles triangles, height to width ratios of 1.6:1 and 1.7:1 were found to be the most pleasant, gathering an equal number of votes of preference from the 250 subjects tested.
- Another experiment was conducted with 12 crosses, in which 140 subjects took part. The cross with a crossbeam one half of the length of the vertical bar with the crossbeam placed so that one quarter of the vertical bar remained on the top was found to be most aesthetically pleasing.

3.3.4 Other Studies and More Recent Research

The following is a summary of several other relevant studies carried out on the aesthetic value of GR in the perception of visual forms:

- R. Angier published the findings of a study in 1903 on the aesthetics of unequal divisions of horizontal lines [15]. He concluded that aesthetic preference for certain ratios is mainly characterized by individual differences in the experimental method and that any measures of central tendencies calculated to represent such preferences are only artifacts of the process, which obscure more than they reveal.
- P. Farnsworth detailed an experiment in 1932 using the paired comparison of nine rectangles presented in both vertical and horizontal orientation [16]. He found most preferences for those with an aspect ratio close to GR for vertical orientation and 1.5 for horizontal orientation.
- W. C. Shipley, P. .E. Daltman, and B. A. Steele detailed an experiment in 1947 in which the subjects ranked their preference for two series, each containing six rectangles [17]. One series had rectangles with a constant area, the other had rectangles of constant length. An average preference for the rectangle in the series with an aspect ratio of 1.54:1 was determined.
- C. W. Nienstedt and S. Ross concluded in 1951 in an experiment with 100 college students that both the number of first choices and the median rank of preferences was near GR for the aspect ratio of rectangles [18]. The preference of a group of significantly older subjects was determined to be approximately 1.33.

- M. Godkewitsch concluded in a 1974 article detailing an experiment involving rectangles with 165 subjects that the preference for GR "is an artifact of its position in the range of stimuli presented and of the measures of preference rather than of any intrinsic aesthetic quality," noting (as Witmer also suggested in his studies) that subjects often tended towards a middle of the stimulus series [19]. The studies of Fechner and Lalo among others, however, call this conclusion into question. According to Godkewitsch, "Aesthetic theory has hardly any rationale left to regard the golden section as a decisive factor in formal visual beauty."

- In an article appearing in 1976, J. Benjafield detailed the results of an experiment he had carried out with 180 subjects, in part in response to what he viewed as a shortcoming in Godkewitsch's experiment – the fact that no control was present for the size of the rectangles, and that size and proportion were confounded [20]. In Benjafield's experiment, he compared the judgments on two series of rectangles: one in which the area of the rectangles varied according to the ratio of length to width, and another in which all of the rectangles were of equal area. In the former series, the subjects tended to select the larger rectangles, whereas in the latter the subjects tended towards rectangles approximating GR.

- H. R. Shiffman and D. J. Bobko detailed in an article published in 1978 an investigation into the division of lines at different angles and found great discrepancies between individuals as well as for the same individual with the line at different angular positions [21].

- C. Plug published an article on GR in 1980 in which he includes a detailed summary of research carried out up until that point [22]. He also details an experiment with 523 subjects, none of which had previously taken part in any experiments involving form perception. The main purpose was to determine whether the subjects drawing certain forms were be influenced by the presentation of biased sets of shapes just before they drew their own. Subjects were also asked to rank their preference of seven diamonds beforehand. There were three groups of seven diamonds presented to different groups of subjects. These three groups overlapped in their range of length-to-width ratios, but the subjects nonetheless were found to tend towards the extremes of the group they were presented with (contradicting the assertions of Godkewitsch), thus discounting the aesthetic relevance of the actual length-to-width ratios. When the drawn figures were measured, no influence was found of the biased groups of seven diamonds the subjects had seen before. A median length-to-width of 1.95:1 was determined.

- H. Höge detailed two experiments he had carried out in an article published in 1996 [23]. In the first experiment, 31 subjects were given four basis lines of lengths 3.1, 5.1, 7.2 and 9.2 centimeters. This experimental group was told to use these lines as part of four separate rectangles that they were supposed

to create to be *beautiful*. A control group of 31 other subjects was given the same task but without being told any specific instructions regarding beauty. No preference for GR was found. In a second experiment, two separate random samples of 88 rectangles from the first experiment (each consisting of 44 from the experimental group and 44 from the control group) were created. Ten subjects analyzed the first sample and ten the second sample. They were told to attempt to guess which one was from the experimental group (i.e. which rectangles were created specifically to be beautiful) and which were from the control group. Höge found it interesting that subjects sorted significantly more rectangles into the *beautiful* category from the experimental group than from the control group. No preference for GR was found, however, in the resulting group that the subjects guessed were designed with beauty in mind.

3.4 The Research Problem

It is part of human nature to desire to understand what we find to be beautiful and why.[5] Considering the fact that significant interest has been taken in conducting experiments designed to determine the aesthetic value of GR in the visual domain, it is surprising that no parallel experimental research has ever been carried out in the audio domain. Even the earliest researchers covered here, Fechner and Zeising, had speculated about the potential value of FS and GR in music (see Appendix B). Perhaps the main reason for this complete absence of parallel research has been a lack of suitable example stimuli for study, or a lack of understanding of how such stimuli would be constituted; as Fechner indicated in his writings, such studies necessitate simple examples in order to limit both associative and combinatorial influence. Considering this lack of research in the audio domain, we can formulate our research question in this dissertation as follows: **Do sonifications of mathematical objects closely related to FS and GR hold some special aesthetic significance or advantage over less closely related sonified mathematical objects?**

In order to answer this question, we will first undertake a thorough historical survey of works in which composers utilized FS or GR. Trends in the utilization of FS and GR will be identified for the first time and new terminology will be introduced. In addition to describing the composers' methods of utilization, the analyses will show an abundance of combinatorial and associative influences present in such works, which render them practically useless in the scientific study of the aesthetic value of FS and GR. Such works were simply never intended for research purposes. Following this historical survey, criteria for a method of direct sonification will be outlined that

[5]The field of evolutionary aesthetics is concerned with understanding humans' aesthetic judgment and their ability to recognize beauty as a biologically adapted ability to make important decisions [24]. Stephen Davies suggests three possible evolutionary roles for art and aesthetics [25]. First, they could be adaptations that have increased the fitness of those who possessed them. Second, they could be spandrels – byproducts of the evolution of some other characteristic – rather than a direct product of adaptive selection. Lastly, they could be vestiges – features that were functional for precursors of a species but which are currently non-functional. A fourth possibility is the null hypothesis that there is no significant connection between art and evolution.

can be useful in mitigating such influences and creating the simple example stimuli necessary for research. After a presentation of various sonifications by the author, the results of a novel experiment in the aesthetic value of FS and GR in the audio domain carried out at UCSB will then be detailed. In this experiment, sonifications are compared for the first time in history for their relative aesthetic value in a scientific context.

3.5 Mathematicians' Interest in FS and GR

FS and GR have attracted a great deal of attention among integer sequences and ratios. As mentioned in 1.1 and 1.2, GR has been studied in diverse cultures at least since approximately 300 BCE, and FS since at least 800 CE. GR is sometimes called "the most irrational number" [26]. Given the rational approximation p/q of an irrational number R with integers p and q, the measure of the degree of irrationality of R can be determined from the error in approximating R with p/q. As p and q increase, the error decreases. GR is the number for which the error decreases at the slowest rate as p and q increase [27]. An entire mathematical journal, *The Fibonacci Quarterly*, has been devoted to the study of Fibonacci-related mathematics since 1963. According to personal correspondence with members of the editorial board and multiple sources, the journal has had a continuous backlog of articles since its inception [28, 29, 30]. The *Fibonacci Association*, also founded in 1963, holds biannual international conferences that are regularly attended by mathematicians from every continent. Hundreds of books and thousands of articles have been published specifically on mathematics related to FS and GR.

3.6 General Scientific Interest in FS and GR

Zeising's assertion that GR formed the basis of a fundamental morphological law in nature was viewed with great skepticism by scientists during his lifetime. Although the scope of such a law may have been exaggerated by Zeising, there are some concrete, undisputed examples of occurrences of FS and GR in nature. Here we will take a brief look at two phenomena in life science involving plants and animals.

3.6.1 Parastichies

Certain cacti, flowers and conifer cones often display parastichies – hypothetical spiral lines connecting the bases of a series of leaves or seeds on a stem – that are arranged according to GR. In the following image of a pinecone, for example, one can see $F_7 = 13$ spirals in one directions, and $F_6 = 8$ in the other, as marked by the green and red lines in the parallel image on the right.

Figure 6: Parastichies on pinecones.



The two images below are of a cactus and a sunflower. In the image on the left, there are $F_7 = 13$ spirals in one direction and $F_6 = 8$ in the other. On the seed head of the sunflower, we see parastichies numbering $F_9 = 34$ in one direction and $F_8 = 21$ in the other.

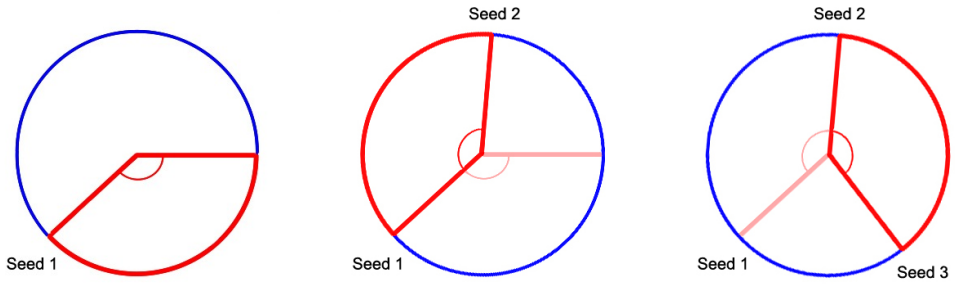
Figure 7: Parastichies on a cactus and sunflower.



$\varphi * 2\pi$ radians (approximately 137.51°) is sometimes referred to as the golden angle. Each successive seed (or glochid on a cactus) in this morphological pattern emerges at an angle $\varphi * 2\pi$ radians from the previous, providing for the most uniform possible placement [31]. The following series of images illustrates the principle of seeds

emerging according to the golden angle:

Figure 8: Seed placement according to the golden angle.

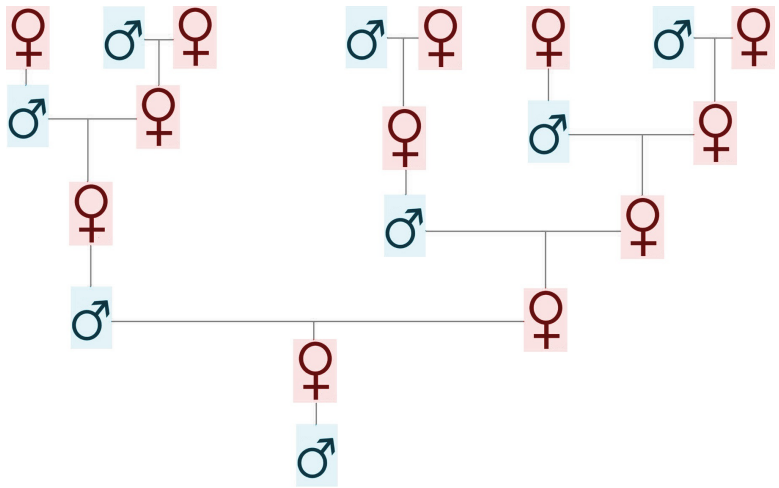


As each successive seed emerges from the center, the previous seeds are pushed outward. According to Mario Livio [32], some early observations of this phenomenon were made by Charles Bonnet (1754) [33]. More extensive research was then carried out by Alexander Braun (1835) [34], Carl Friederich Schimper (1836) [35], and Auguste and Louis Bravais (1839) [36].

3.6.2 Bee Genealogy

If one looks at the family tree of a single bee, one finds Fibonacci numbers of ancestors in each generation [26]. This is true because every female bee has both a mother and a father, whereas a male bee only has a mother, having been produced by an unfertilized egg from a queen. The following diagram shows the genealogy of both male and female bees:

Figure 9: FS in bee genealogy.



There are F_n bees in each generation: F_{n-1} females and F_{n-2} males. The binary sequence of males and females obtained by taking the limit of the genealogical tree above $< ♀, ♂, ♀, ♀, ♂, ♀, ♂, ♀, \cdots >$ is known as the rabbit sequence or golden string. Later we will make use of this Fibonacci-related sequence in sonifications as well as in our experiment.

3.7 FS and GR in Popular Culture

FS and GR have been assimilated into popular culture in various guises, particularly in the United States in the last two decades, demonstrating a wider appeal to the general public in addition to the interest taken in them by scientists, mathematicians and artists. The following are just a few examples of such assimilation:

- GR is featured in Walt Disney's educational featurette *Donald in Mathmagic Land* (1959) [37]. Donald learns how GR was (supposedly) utilized in ancient Greek architecture.
- The word *Fibonacci* has been incorporated into the names of several musical ensembles: The Fibonacci Sequence (a chamber ensemble founded in 1994 in the UK); the Trio Fibonacci (a Montreal chamber ensemble established in 1998); Fibonacci Sequence (a progressive rock band founded in 2008 based in Milwaukee, WI).
- Both FS and GR feature prominently in Darren Aronofsky's psychological thriller π (1998) [37].
- FS was used in creating the lyrics of the American hip hop duo Black Star's song *Astronomy (8th Light)* (1998).
- FS was used in the syllable patterns in the lyrics of the American rock band Tool's song *Lateralus* (2002).
- FS appears in various parts of Dan Brown's novel *The Da Vinci Code* (2003) [38]. The novel was later adapted to film and released in 2006 by Columbia Pictures.
- Blogger Gregory K. Pincus introduced the concept of a poem containing at least six lines in which each successive line has a Fibonacci number of syllables in a blog post from April 1, 2006. He calls such a poem a *Fib*. Many bloggers followed suit, posting their own Fibs [39].
- FS is featured in the episodes entitled *Safe* and *The Same Old Story* of the American science fiction television show *Fringe* (2008–2013).
- The October 11, 2005 edition of Bill Amend's comic strip *FoxTrot* features a character snapping an American football using the Fibonacci sequence when giving the count. The February 8, 2009 edition features characters eating Fibonacci numbers of nachos.

4 GFS and GR in the Music of the Twentieth and Twenty-First Centuries

Innumerable composers of the twentieth and twenty-first centuries have used GFS or GR in their music in various ways. Certain trends have emerged that have not yet been discussed in the literature. In order to gain some insight into what role the conscious application of GFS and GR has played and what reasons for their utilization have emerged, we shall take a number of pieces where the composer is known to have consciously used them and analyze them. Both well-known and lesser-known composers' works find representation in this section. In the course of our analysis, will see that all of the examples are characterized by a level of complexity that would exclude them from possible utilization in experiments designed to determine the aesthetic value of GFS and GR (due to the presence of an abundance of combinatorial and associative influences). Before proceeding, however, we will take a look at some of the ways that GFS can be applied in composition in order to develop a framework for analysis.

4.1 Applications of GFS to Musical Parameters

As mentioned, composers have applied GFS and GR in their compositions in various ways. A direct application of GR is rare, however, as we will see. This can perhaps be attributed to the fact that irrational numbers – whether applied to pitch, rhythm, meter, etc. – generally cannot be realized with great accuracy by performers on traditional instruments, or as Xenakis put it [40]:

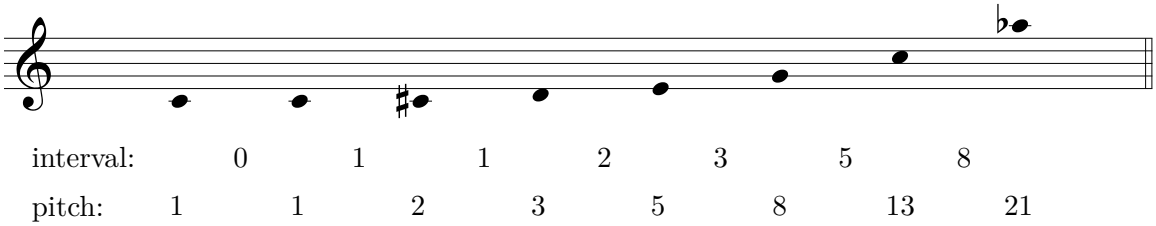
> The golden number is irrational in mathematics. In music it is not possible (exception: electronic music). Providing durational values that a normal musician can count, I formed a [Fibonacci] sequence as just described.

We will therefore concentrate on the traditional applications of GFS in developing our framework, as they will prove most useful in analysis. Those applications that do not fit into this framework can be considered anomalies and particularly unique. In developing this framework, it shall not be our purpose to exhaust all possible applications, but rather to shed light onto those that are most common as well as those that seem most intuitive. We will see that, in practice, these applications are intermixed with various other compositional techniques that have nothing to do with GFS.

4.1.1 Pitch

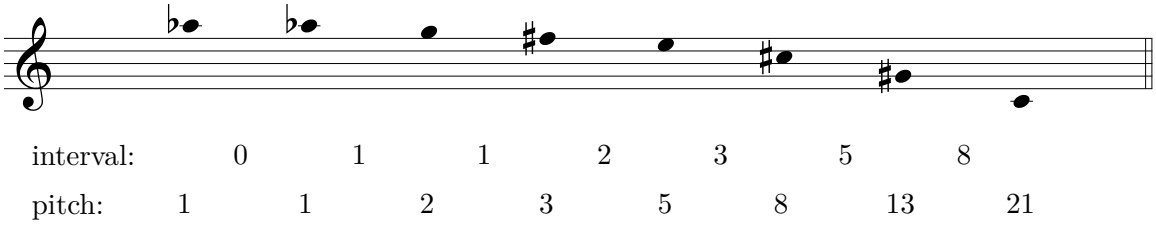
We can create a sequence of pitches – or equivalently, a sequence of intervals – based on FS in twelve-tone equal temperament:

Figure 10: Fibonacci pitch sequences (ascending).



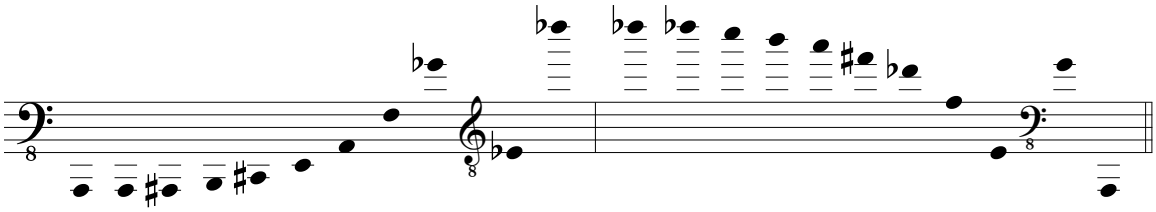
This could be inverted, and we could start on a different pitch:

Figure 11: Fibonacci pitch sequences (descending).



The primary challenge composers face is that the pitches very quickly exceed the range of our hearing; this issue is not unique to any particular application and is due to the additive nature of FS.[6] The following example shows the largest possible audible melodic sequence of intervals based on FS in twelve-tone tuning $< 0, 1, 1, 2, 3, 5, 8, 13, 21, 34 >$:

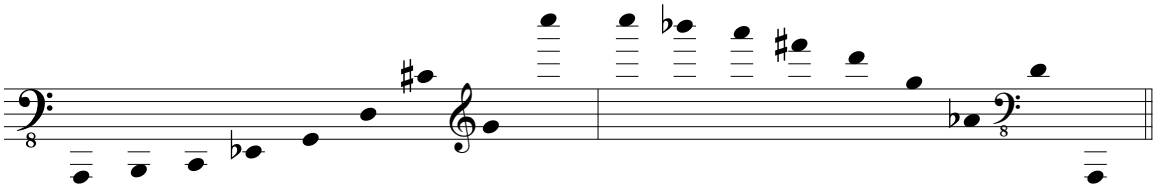
Figure 12: Largest Fibonacci pitch sequence.



[6]Even if we started with extremely small intervals – and even if it took a while for that reason – as soon as our ear could discriminate an interval, from that point it would very quickly escape our hearing range.

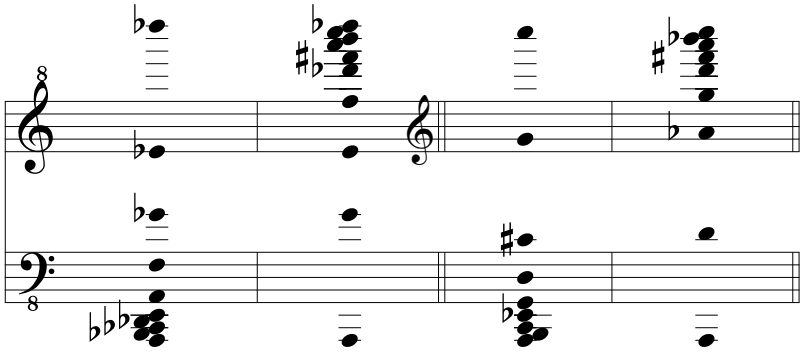
The largest possible LS in twelve-tone equal temperament, ascending and descending $< 2, 1, 3, 4, 7, 11, 18, 29 >$:

Figure 13: Largest Lucas pitch sequence.



These were melodic applications of *Fibonacci pitch sequences*. The same sequences can be expressed harmonically:

Figure 14: Fibonacci pitch sequences expressed harmonically.



This is the largest range these sequences can span in this tuning.

Some composers have used a modulus and pitch class equivalence, creating what are called *Fibonacci pitch class sequences*, using the following equation:

$$X_n \equiv X_{n-1} + X_{n-2} \pmod{m}.$$

In this way, we lose the property that each interval is the sum of the previous two, but are free to take advantage of octave equivalence as we wish, and the property that each interval or pitch class is the sum of the previous two is retained. If we exhaust all possible GFS sequences in twelve-tone tuning, the result is a series of ten periodic sequences. In the following table, the sequences have been truncated to the length of their period:

Figure 15: Complete residue system modulo 12.

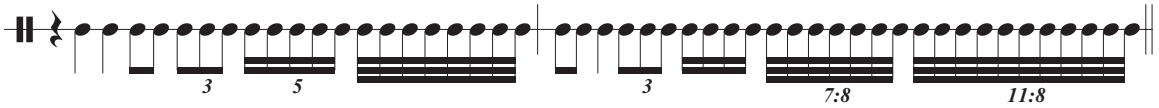
0																							
0	1	1	2	3	5	8	1	9	10	7	5	0	5	5	10	3	1	4	5	9	2	11	1
0	11	11	10	9	7	4	11	3	2	5	7	0	7	7	2	9	11	8	7	3	10	1	11
0	2	2	4	6	10	4	2	6	8	2	10	0	10	10	8	6	2	8	10	6	4	10	2
0	3	3	6	9	3																		
0	9	9	6	3	9																		
0	4	4	8	0	8	8	4																
0	6	6																					
2	1	3	4	7	11	6	5	11	4	3	7	10	5	3	8	11	7	6	1	7	8	3	11
10	11	9	8	5	1	6	7	1	8	9	5	2	7	9	4	1	5	6	11	5	4	9	1

This is sometimes referred to as a complete Fibonacci system modulo 12 [41]. Every possible ordered interval class and pitch class dyad is contained once and only once in it. In other words, any two integers between 0 and 11 will be found next to one another in this table once and only once.[7] In addition to application to pitch, these sequences could be treated in a serial manner and applied to various other musical parameters as well.

4.1.2 Rhythm

Some composers have applied GFS to rhythm. Two general possibilities exist, the first of which is called *Fibonacci attack summation*, in which, given two starting values, each beat contains a number of equally spaced attacks equal to the number of attacks in the previous two beats. We could sonify $< 0, 1, 1, 2, 3, 5, 8 >$ and $< 2, 1, 3, 4, 7, 11 >$, for example, as such:

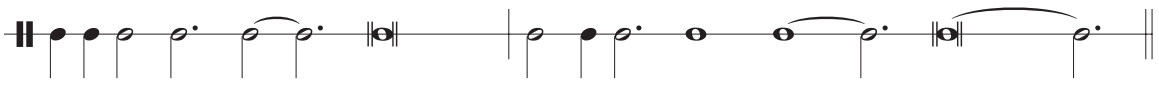
Figure 16: Fibonacci attack summation.



[7] In some cases (e.g. $< 1, 0 >$ or $< 11, 2 >$), due to periodicity and the truncation in this table, it will be necessary to wrap around in the table from the last integer in a sequence to the first.

The second technique, called *Fibonacci pulse summation*, is the inverse of the above: each attack contains a number of pulses equal to the number of pulses attributed to the previous two attacks. Put another way, each note value is equal to the sum of the previous two. The latter definition might be more relevant for sustaining instruments, the former for those with a shorter attack. $< 0, 1, 1, 2, 3, 5, 8 >$ and $< 2, 1, 3, 4, 7, 11 >$ could be sonified, for example, as such:

Figure 17: Fibonacci pulse summation.



4.1.3 Meter

Another temporal application of GFS directly related to Fibonacci pulse summation is *metric summation*, in which, given two starting values, each measure contains the number of beats contained in the previous two. We could create a series of meters based on FS, for example, as such:

Figure 18: Fibonacci metric summation.



LS might be used as well:

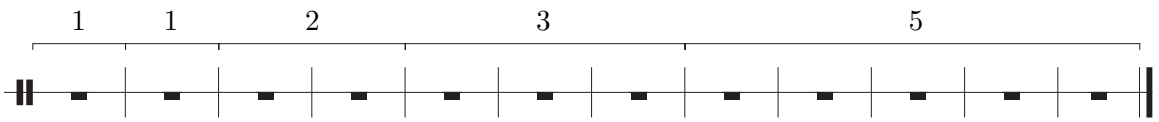
Figure 19: Lucas metric summation.



4.1.4 Proportion and Form

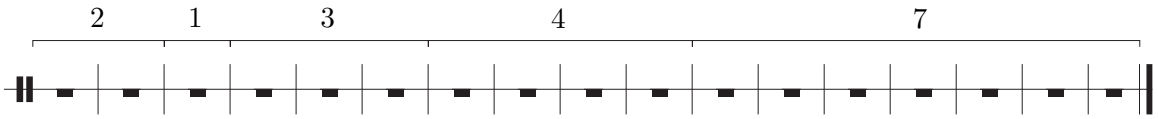
One can divide up a work formally according to FS, so that sections are defined in length according to the sequence, as such:

Figure 20: Application of FS to formal proportion.



Or according to LS:

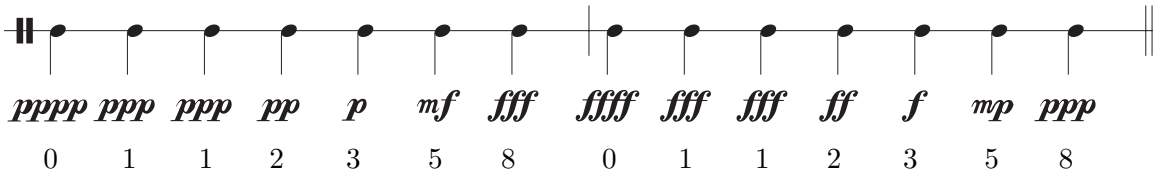
Figure 21: Application of LS to formal proportion.



4.1.5 Dynamics

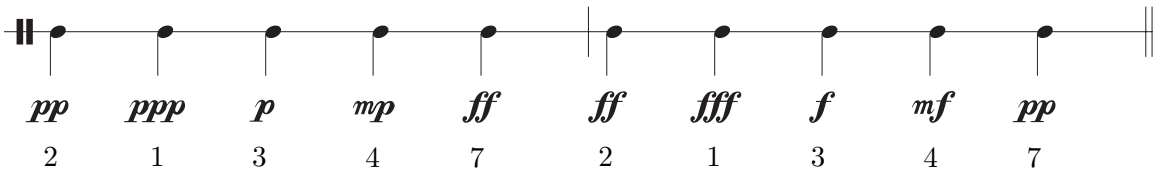
GFS might also be applied to dynamics. FS $< 0, 1, 1, 2, 3, 5, 8 >$, for instance, could be applied to dynamics in the following two ways, either increasing in loudness, or decreasing:

Figure 22: Application of FS to dynamics.



LS $< 2, 1, 3, 4, 7 >$ might appear as such:

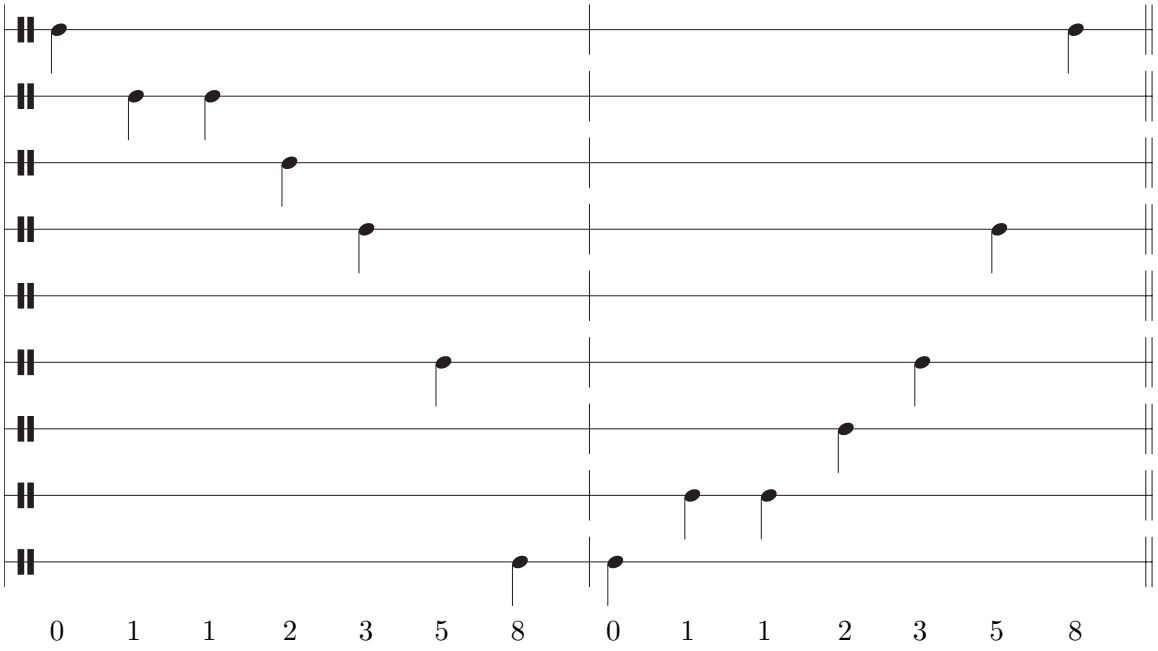
Figure 23: Application of LS to dynamics.



4.1.6 Spatial Location

Mapping GFS to spatial location is possible as well; $< 0, 1, 1, 2, 3, 5, 8 >$ might emanate from different locations. If nine players (representing the integers 0 through 8) were lined up equidistantly, FS could be applied as such:

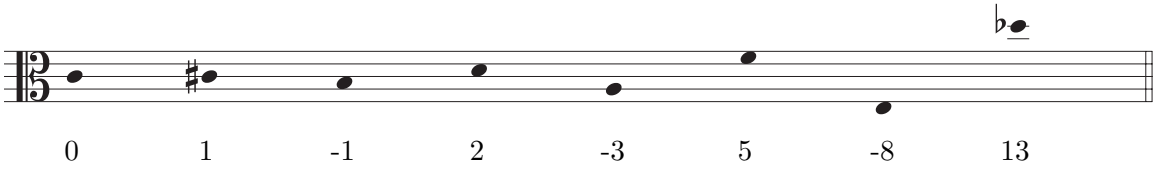
Figure 24: Application of FS to spatial location.



4.1.7 Inverse GFS

As mentioned earlier, an inverse GFS is a sequence in which $G_n = G_{n-2} - G_{n-1}$. If the starting values are again $G_0 = 0$ and $G_1 = 1$, then the sequence begins $< 0, 1, -1, 2, -3, 5, -8, 13, \cdots >$. This could be sonified in pitch, for example, as follows:

Figure 25: An inverse Fibonacci pitch sequence.



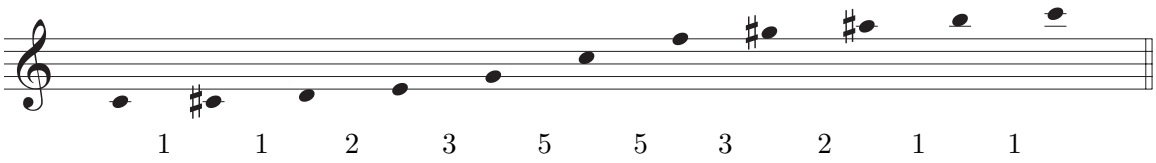
Applying negative numbers in the time domain does not make sense for our purposes, but we could potentially apply inverse GFS to some other parameters such as dynamics and location as well in a similar manner as has been demonstrated.

4.1.8 Reversal

Many composers have used a technique called *reversal* in which, at a certain point, they reverse their direction in the sequence, utilizing an inverse GFS. Reversal can take different forms depending on the musical parameter GFS is being applied to. In

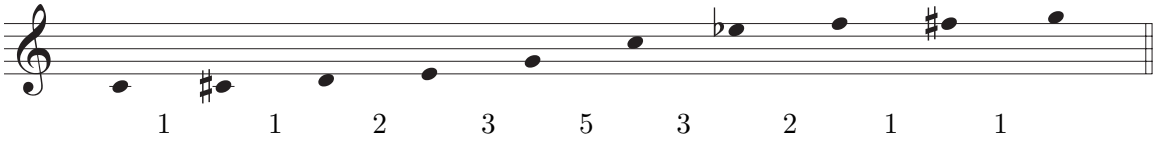
the pitch domain, several possibilities exist. The first is called *reversal with repetition*, because the pivot interval (in this case 5) is repeated:

Figure 26: Reversal with repetition.



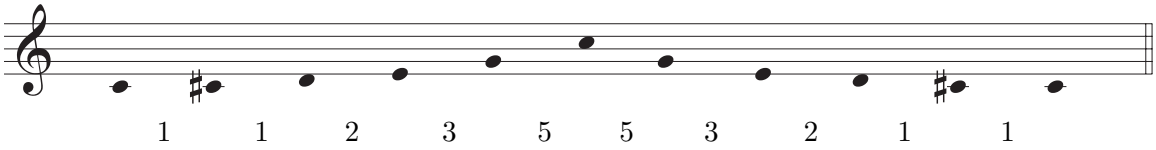
The second, *reversal without repetition*, might take the following form:

Figure 27: Reversal without repetition.



In the pitch domain, it is also possible to simply switch the direction from ascending to descending, or vice versa. This is called *reversal with change of direction*:

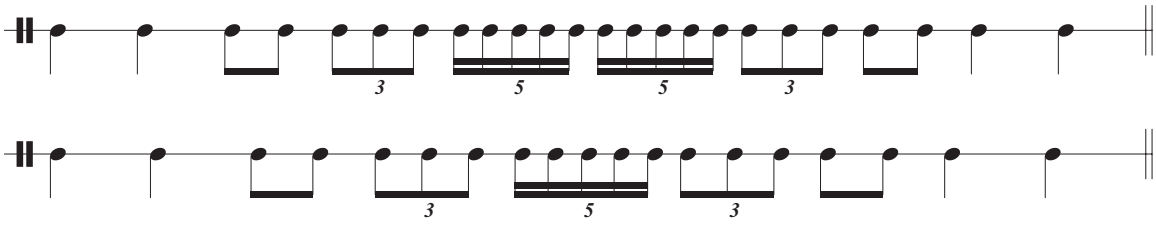
Figure 28: Reversal with change of direction.



In the temporal domain, no change of direction is possible: we cannot go backwards in time. There are thus two main possibilities for reversal in metric and rhythmic applications: reversal with repetition and reversal without repetition.

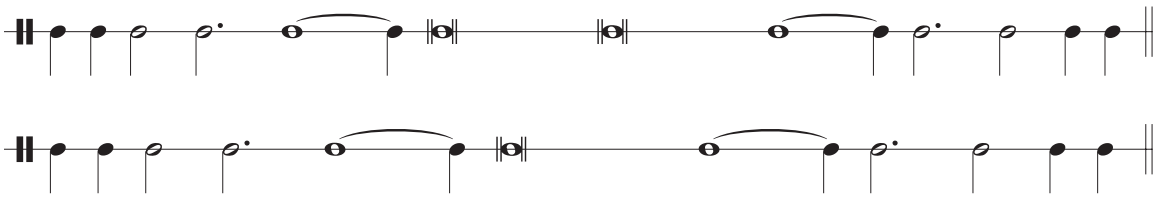
Reversal with attack summation, with and without repetition:

Figure 29: Reversal with attack summation.



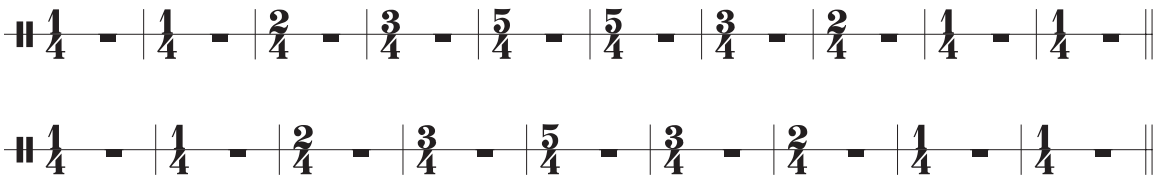
Reversal with pulse summation, with and without repetition:

Figure 30: Reversal with pulse summation.



Reversal in the metric domain, with and without repetition:

Figure 31: Reversal in the metric domain.



Reversal can be utilized in almost any domain GFS can be applied.

4.2 A Survey of Works Utilizing GFS and GR

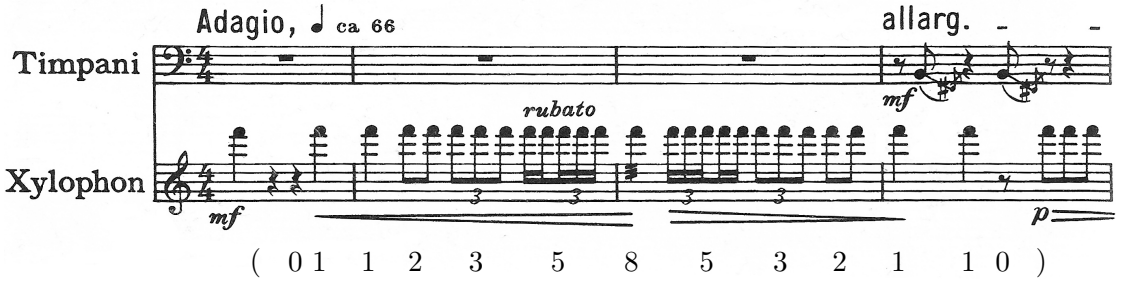
We will now undertake brief analyses of the works of a number of different composers. Our primary goal will be to gain some insight into the variety of ways in which GFS and GR have been applied and to identify trends that have emerged. We will proceed in chronological order. In order to avoid making whether or not the composer utilized GFS and GR the subject of these analyses, we will only analyze works where it is obvious or known that a conscious application has taken place.

4.2.1 Béla Bartók's *Music for String Instruments, Percussion and Celesta* (1936)

One of the earliest apparent conscious applications of FS occurs in the third movement of Bartók's *Music for String Instruments, Percussion and Celesta* [42].[8] We could consider the rhythm in the first four measures of the movement to be a form of Fibonacci attack summation with reversal, although the beat in the second measure that contains five attacks does not use a quintuplet, but two sixteenths followed by three triplet sixteenths:

[8]Although the first movement of *Music for String Instruments, Percussion and Celesta* has been analyzed by multiple authors in terms of proportions based on FS and GR, Roy Howat has shown that there are significant inaccuracies in such analyses [43].

Figure 32: Bartók's *Music for String Instruments, Percussion and Celesta.*



The indication of *rubato* at that point makes it clear that Bartók did not want this rhythm to be executed with perfect mechanical precision. The 0 on the final beat of the pattern (the third beat of the fourth measure) is not represented by a full quarter rest, but by an eighth rest; the tempo, however, has slowed by that point, so that it might be considered to be the final integer of the mirrored pattern. Multiple authors have suggested that Bartók used FS and GR in other works as well [44] [45]. Unfortunately, Bartók is not known to have ever discussed why he utilized FS in composition.

4.2.2 The Contributions of Joseph Schillinger (1941)

Although the author knows of no practical work in which Joseph Schillinger utilized GFS, his theoretical work would be too important not to mention here. In his book *The Schillinger System of Musical Composition* [46], published in 1941 posthumously, he gives examples of GFS applied to pitch with octave transpositions, as seen below:

Figure 33: FS and LS applied to pitch with octave transpositions.

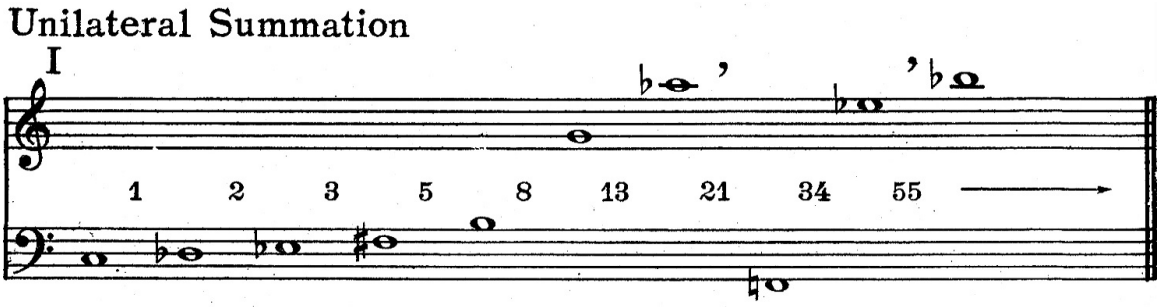


Figure 106. Unilateral symmetry of Fibonacci series (concluded).

Figure 33: FS and LS applied to pitch with octave transpositions (continued).

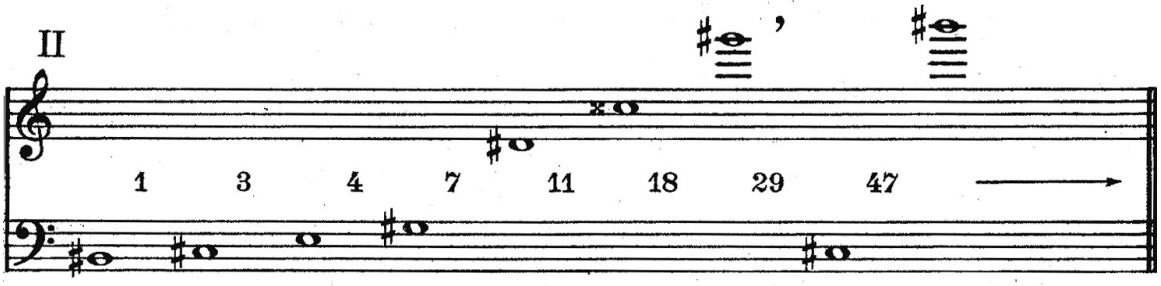


Figure 108. Second summation series applied to melody.

He stresses the symmetric nature of FS:

> A property of the summation series known as Fibonacci series [sic] is that it contains symmetry throughout. The word *symmetry* emphasizes the equality of two measured ratios, according to an authority on the subject, Dr. William Churchill.

He also mentions inverse Fibonacci pitch sequences as well, which he describes as being characterized by bilateral symmetry.

Figure 34: Bilateral symmetry in Fibonacci pitch sequences.

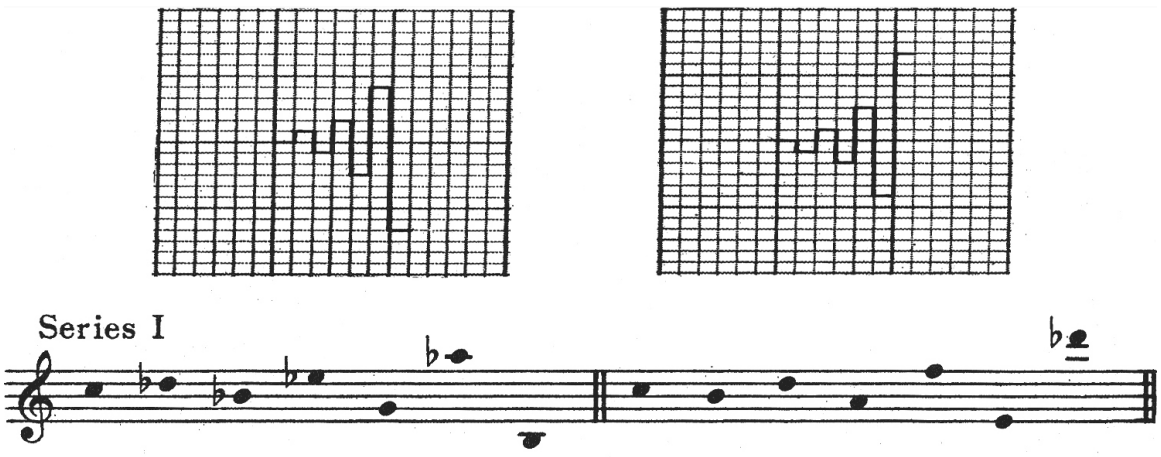


Figure 112. Bilateral symmetry in first summation series.

We note that his notation does not reflect a perfect inverse FS as in his diagrams; he appears to have accidentally skipped the interval 2 and extended the final interval to 21 in his notation.

We can gain some further idea of why Schillinger was interested in FS and GR from the following quote from the mentioned book, referring to motifs based on summation sequences:

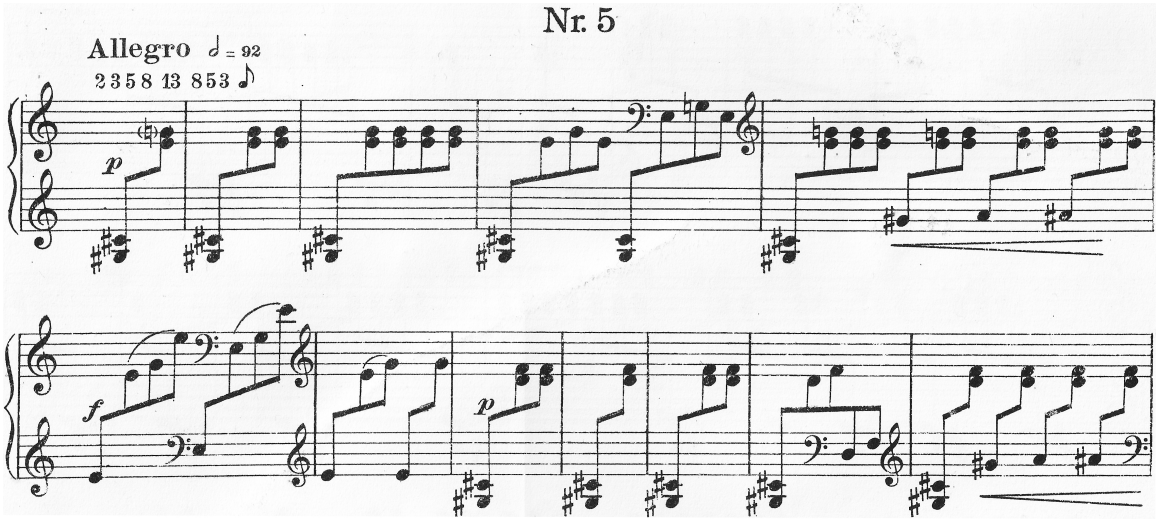
> These motifs have such a universal appeal that, whether they appear in folk music or in the work of an individual composer, they become universally accepted as definite crystallized symbols of musical expression.

In *The Mathematical Basis for the Arts* [47], a work whose "goal is to disclose the mechanism of creatorship as it manifests itself in nature and in the arts," he also gives examples of GFS in his chapter *Continuity*. In that book, however, he does not suggest an application of the sequences to any specific parameters, leaving them instead as raw compositional material.

4.2.3 Boris Blacher's *Ornamente* (1950)

Boris Blacher's piano composition *Ornamente* [48], consists of seven studies in variable meters. In each study, the meter changes every measure according to a certain sequence. In the fifth study, dedicated to Rolf Liebermann, he utilizes FS. The piece consists of 113 measures. We observe Fibonacci metric summation and reversal without repetition in the sequence of meters, which consists of groups of $< 2, 3, 5, 8, 13, 8, 5, 3 >$ eighth notes repeated 14 times, followed by a final 2/8 meter that completes the symmetry. The piece begins as follows:

Figure 35: Boris Blacher's *Ornamente*.



We gain some insight into why Blacher applied FS and other sequences in these studies from the following quote from the score:

> The idea of designing the metric process in such a way that every measure is governed by a different metric structure came from the realization that changes of meter often intensify the formal progression. If one bases the metric proportions on mathematical principles, namely on sequences or on combinatorics – in such a way metric progression is no longer a

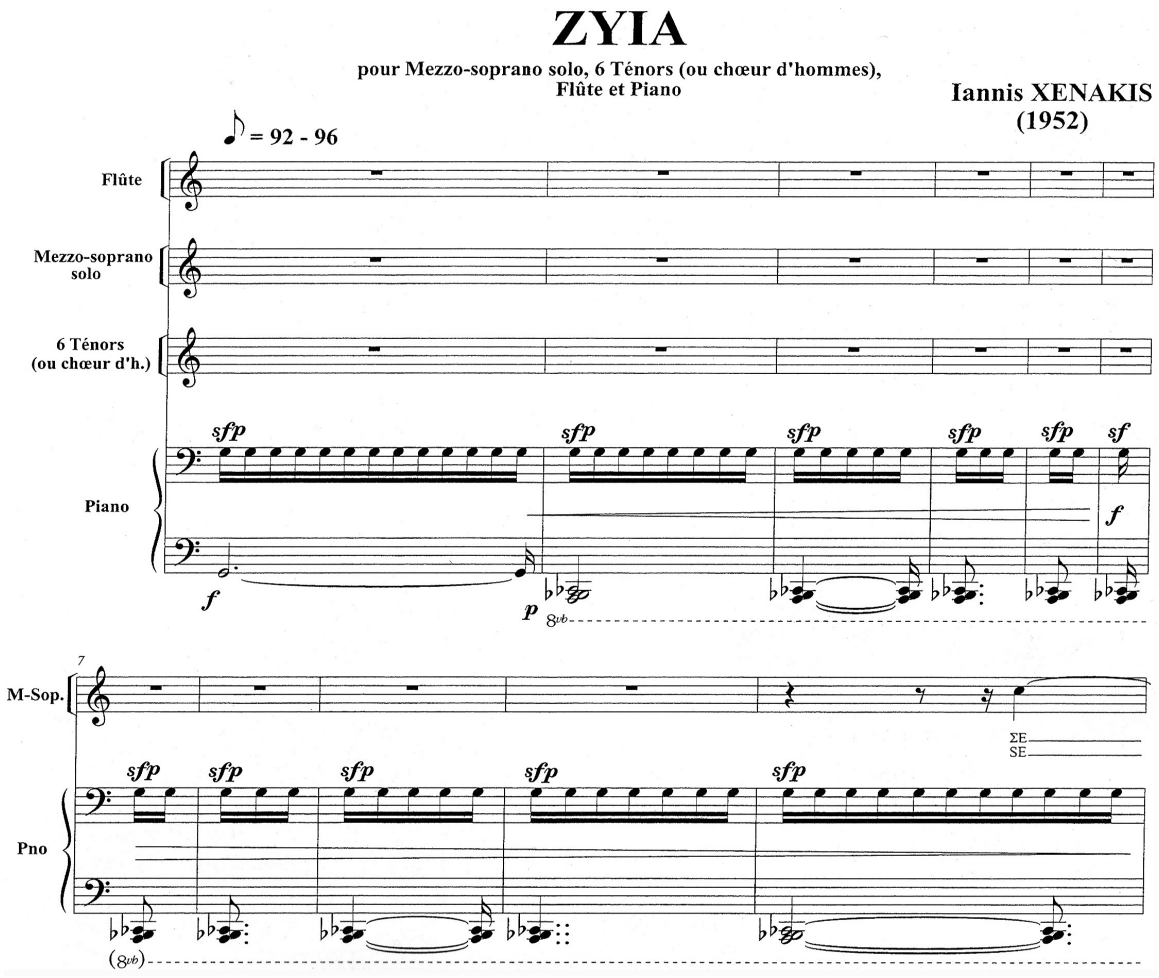
product of capriciousness or randomness. New symmetries on a higher level result, interesting overlaps of metric sequences with musical phrase, variable reprises and other things as well.

Blacher also used FS in his *Musik für Osaka* for the Expo '70 in Osaka, realized with the help of Rüdiger Rüfer. The piece utilized 33 loudspeakers and seven tracks of audio, with spatialization automated through an extra tape track. A description of the work and application of FS can be found online [49].

4.2.4 Iannis Xenakis' *Zyia* (1952)

Xenakis' first explicit application of a mathematical process to music was when he used FS in *Zyia* [50, 51]. The piece opens as follows [52]:

Figure 36: Beginning of Iannis Xenakis' *Zyia*.

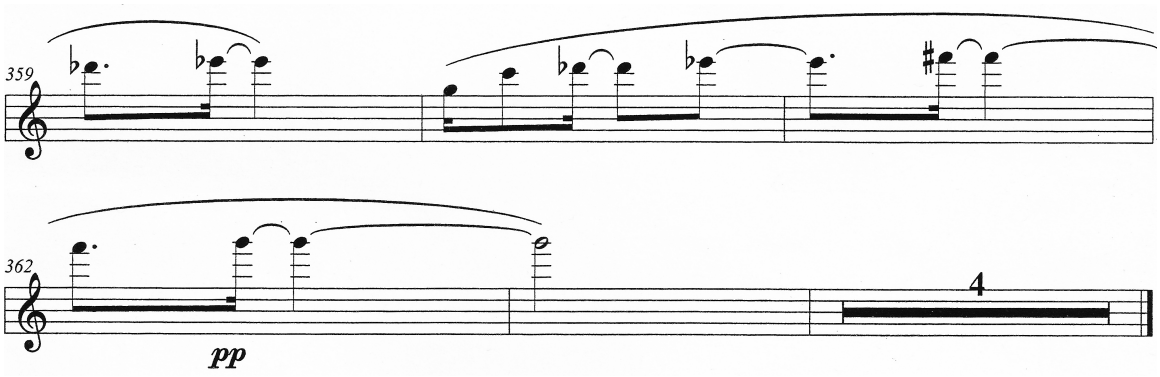


Here, we see an application of inverse Fibonacci metric summation in the pattern of meters in the first nine measures, which contain numbers of sixteenth notes derived from $< 13, 8, 5, 3, 2, 1, 2, 3, 5 >$. In this pattern, we also see the use of reversal without

repetition. One can see that Xenakis breaks the pattern in the tenth measure with a meter containing seven sixteenth notes, shortly after which the soprano enters.

Further application of Fibonacci numbers to individual rhythmic values occurs throughout. At the end of the composition, in measures 360 to 363, the flute pattern utilizes Fibonacci pulse summation in the number of sixteenth notes $< 1, 2, 3, 5, 8, 13 >$ attributed to each note in the ascending phrase:

Figure 37: Conclusion of Iannis Xenakis' *Zyia*.



It has been speculated that Xenakis' interest in FS was sparked by Le Corbusier [53], with whom he was working at the time and who used them in his proportioning system the *Modulor*.

Xenakis discussed his use of FS and GR in *Anastenaria, Le Sacrifice* [40]. In the preface, he states:

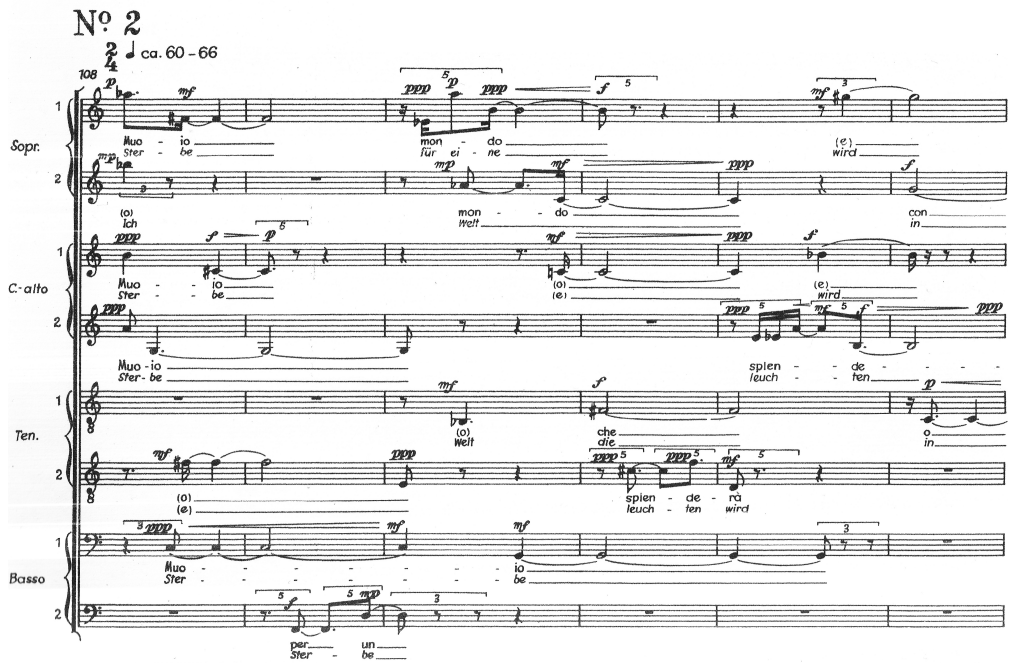
> The golden section is one of the biological laws of growth. One finds it in the proportions of the human body; for example, the ratio of a person's height to the location of the solar plexus is the golden section … musical durations are created by muscular discharges that actuate the human limbs. It is clear that the movements of these limbs have a tendency to correspond temporally to the dimensions of these numbers. Thus the consequence: durations corresponding to the golden ratio are more natural for the movements of the human body … The golden number is irrational in mathematics. In music it is not possible (exception: electronic music). Providing durational values that a normal musician can count, I formed a [Fibonacci] sequence as just described.

These pieces refute the assertions of Charles Madden that Xenakis never used FS and GR in his book *Fib and Phi in Music: The Golden Proportion in Musical Form* [54].

4.2.5 Luigi Nono's *Il Canto Sospeso* (1956)

Luigi Nono utilized FS in the second movement of *Il Canto Sospeso* (for soloists, choir and orchestra) [55]. The text is taken from the farewell letters of resistance fighters condemned to death by the Nazis, and the work is dedicated "a tutti loro" [to all of them]. The first six measures are shown below:

Figure 38: Luigi Nono's *Il Canto Sospeso*.



In this movement, which Jonathan Kramer provides a detailed analysis of [56], continuous and unbroken four-part harmony occurs between the voices. Each of the four parts has its own base pulse value. The four pulse values used are: eighth note, triplet eighth note, sixteenth note, and quintuplet sixteenth. The row form used is the classic wedge: $< 0, 1, 11, 2, 10, 3, 9, 4, 8, 5, 7, 6 >$. There is always only a single row occurring at any one point in time, and the notes of the row are always distributed through the four voices. The row always begins on A and is repeated 19 times. Kramer shows in a diagram how the Fibonacci rhythmic values are serialized and mapped to the pitches in measures 108 to 142, shifting the values of the sequence every time the row begins anew:

Figure 39: Jonathan Kramer's analysis of *Il Canto Sospeso*.

FIBONACCI DURATION SERIALIZATION IN Il Canto Sospeso, SECOND MOVEMENT

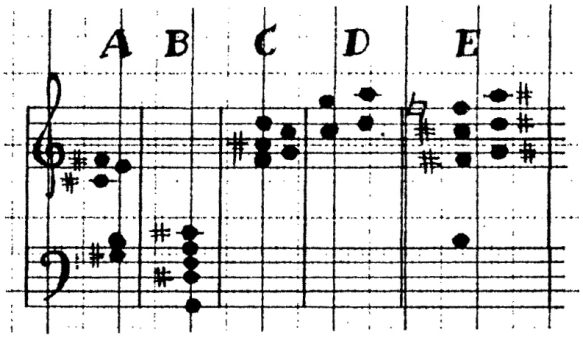
measure in which row form begins:	108	110	112	114	117	119	121	123	125	128	130	132	135	137	139
A	1	2	3	5	8	13	13	8	5	3	2	1	2	3	5
B♭	2	3	5	8	13	13	8	5	3	2	1	1	3	5	8
A♭	3	5	8	13	13	8	5	3	2	1	1	2	5	8	13
B	5	8	13	13	8	5	3	2	1	1	2	3	8	13	13
G	8	13	13	8	5	3	2	1	1	2	3	5	13	13	8
C	13	13	8	5	3	2	1	1	2	3	5	8	13	8	5
F♯	13	8	5	3	2	1	1	2	3	5	8	13	8	5	3
C♯	8	5	3	2	1	1	2	3	5	8	13	13	5	3	2
F	5	3	2	1	1	2	3	5	8	13	13	8	3	2	1
D	3	2	1	1	2	3	5	8	13	13	8	5	2	1	1
E	2	1	1	2	3	5	8	13	13	8	5	*3	1	1	2
E♭	1	1	2	3	5	8	13	13	8	5	3	*2	1	2	3

This is an unusually cryptic serial application of FS, but we can see the use of reversal with repetition in the numbers utilized $< 1, 2, 3, 5, 8, 13, 13, 8, 5, 3, 2, 1 >$.

4.2.6 Juan Amenábar's *Los Peces* (1953–1957)

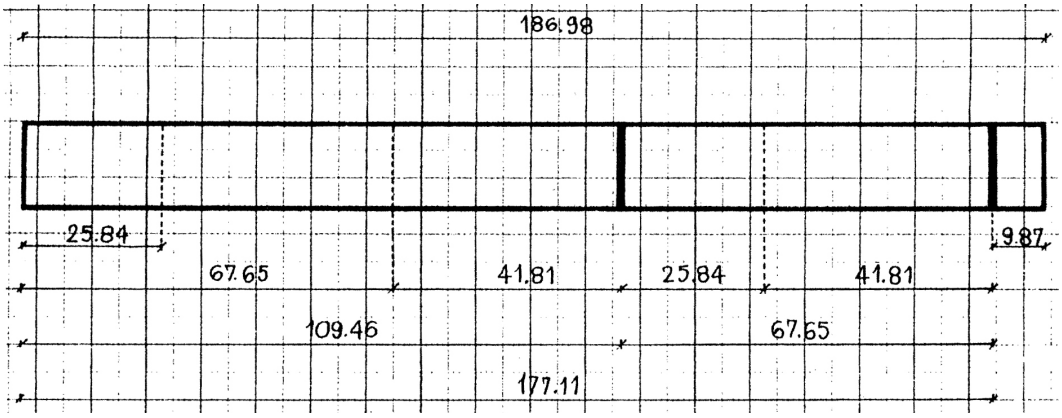
In collaboration with architect Pedro Buchard, Juan Amenábar applied FS to temporal proportion in a tape music piece entitled *Los Peces*, one of the earliest pieces of tape music in Latin America. According to the program notes, provided in an interview by Martín Alejandro Fumarola [57], the pitch material in the work consists of five chords, played on a piano, in which the attacks have been removed:

Figure 40: Pitch material in Juan Amenábar's *Los Peces*.



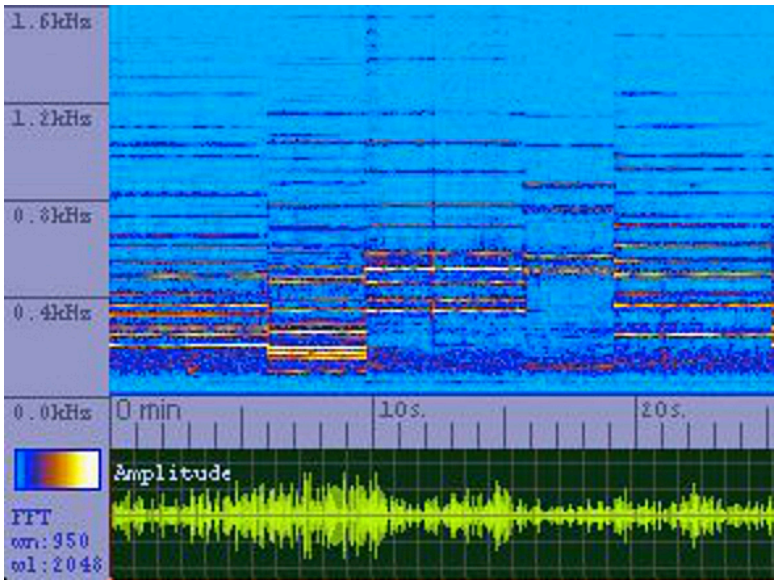
We note, however, that the second chord indicated in these program notes is incorrect: it sounds like the top two notes in the second chord are A-sharp and C, not A and C-sharp. Amenábar provides a glimpse of the metric structure in the program notes as well:

Figure 41: Metric structure of Juan Amenábar's *Los Peces*.



Each square on the graph paper used represents ten seconds. The durations of the sections are derived directly from Fibonacci numbers divided by 100. The first section indicated, for example, lasts 25.84 seconds, which is $F_{18}/100 = 2584/100$ seconds. A spectrogram of the first section of the work, available along with a recording of the work online [58], is shown below:

Figure 42: Spectrogram of Juan Amenábar's *Los Peces*.



The six chords heard in this excerpt, according to the letters given in the program notes, are <A, B, C, C, D, E>; their temporal proportions, or the Fibonacci numbers used, are $< 610, 377, 233, 377, 377, 610 >$, which add up to 2584. Heavy use of reversal with and without repetition is evident throughout and we see some unordered application of FS as well.[9]

Through analog electronic means, Amenábar was able to approximate GR with great precision in this composition.

4.2.7 Ernst Krenek's *Fibonacci Mobile* (1964)

Despite all the author's attempts to get a hold of a score of Ernst Krenek's *Fibonacci Mobile* (for string quartet and piano four hands), he has remained unsuccessful. In an interview by Will Ogdon from 1972 [59], Krenek described his application of FS as follows:

> Another step forward in determinacy has to do with the concept of the mobile. In my orchestra piece *From Three Make Seven* I composed two blocks of music. Of each of these I made three versions at three different speeds, and each of these versions was orchestrated in three different ways (wind instruments, strings, and percussive devices), so that there were eighteen elements to work with, different but closely related to each other. In each element seven time points were indicated at which the elements could be synchronized. The plan of synchronization, the number and sequence of run-throughs and other aspects have to be decided upon before the performance – there is nothing left to improvisation. But the number of possible combinations is enormous.
>
> A similar idea is the basis of my *Fibonacci-Mobile*, a piece for string quartet and piano-four hands. Here the individual elements are proportioned according to a section of the Fibonacci series of numbers: 2 3 5 8 13 21 34, and again the shorter (faster) and longer (slower) elements may be combined in many different ways.

This description implies an unordered temporal application of FS and some aleatoric principles in the organization of the work.

4.2.8 Karlheinz Stockhausen's *Mikrophonie II* (1965)

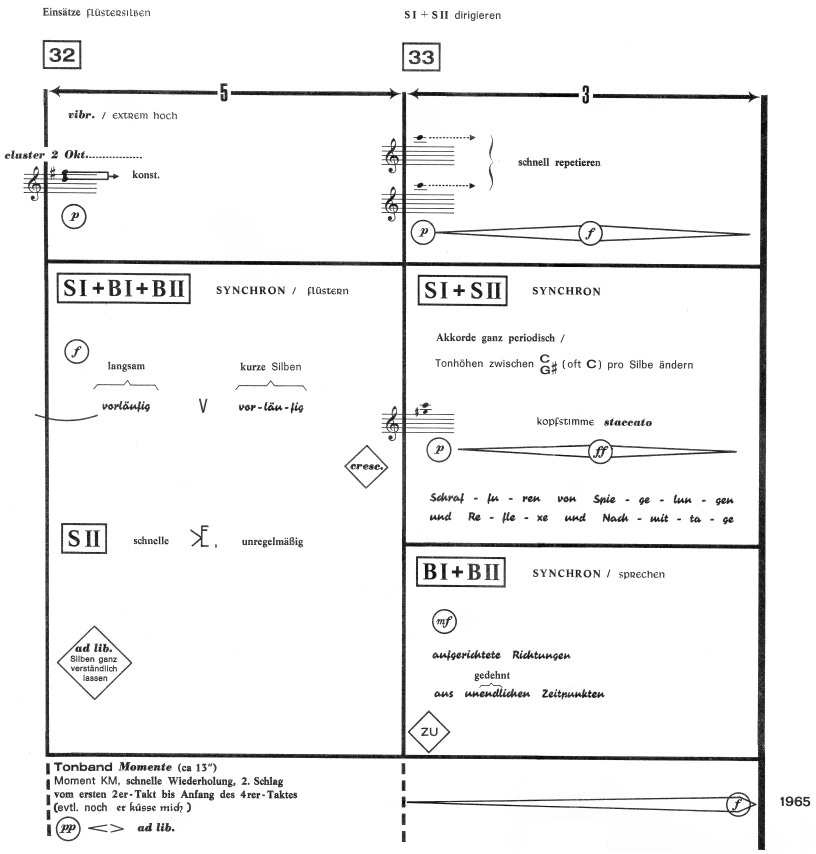
In 1965, Stockhausen finished *Mikrophonie II* (for choir, Hammond organ and four ring modulators), a work in moment form consisting of 33 moments [60]. In this work, an underlying pulse is chosen between 50 and 72 beats per minute that remains constant throughout the composition. FS is applied temporally – not to meter or rhythm, but directly to proportion; each moment lasts a Fibonacci number of pulses. The number of pulses in each of the 33 moments are as follows:

[9] An unordered application of FS can be regarded more as utilization of Fibonacci *numbers* rather than FS itself.

<5,3,144,34,89,8,8,34,34,21,13,13,8,8,21,55,8,55,21,55,13,13,21,13,13,8,8,34,89,8,21,5,3>

As can be seen, this is an unordered application of FS. The last two moments in the composition $< 5, 3 >$ are shown below:

Figure 43: Conclusion of Karlheinz Stockhausen's *Mikrophonie II*.



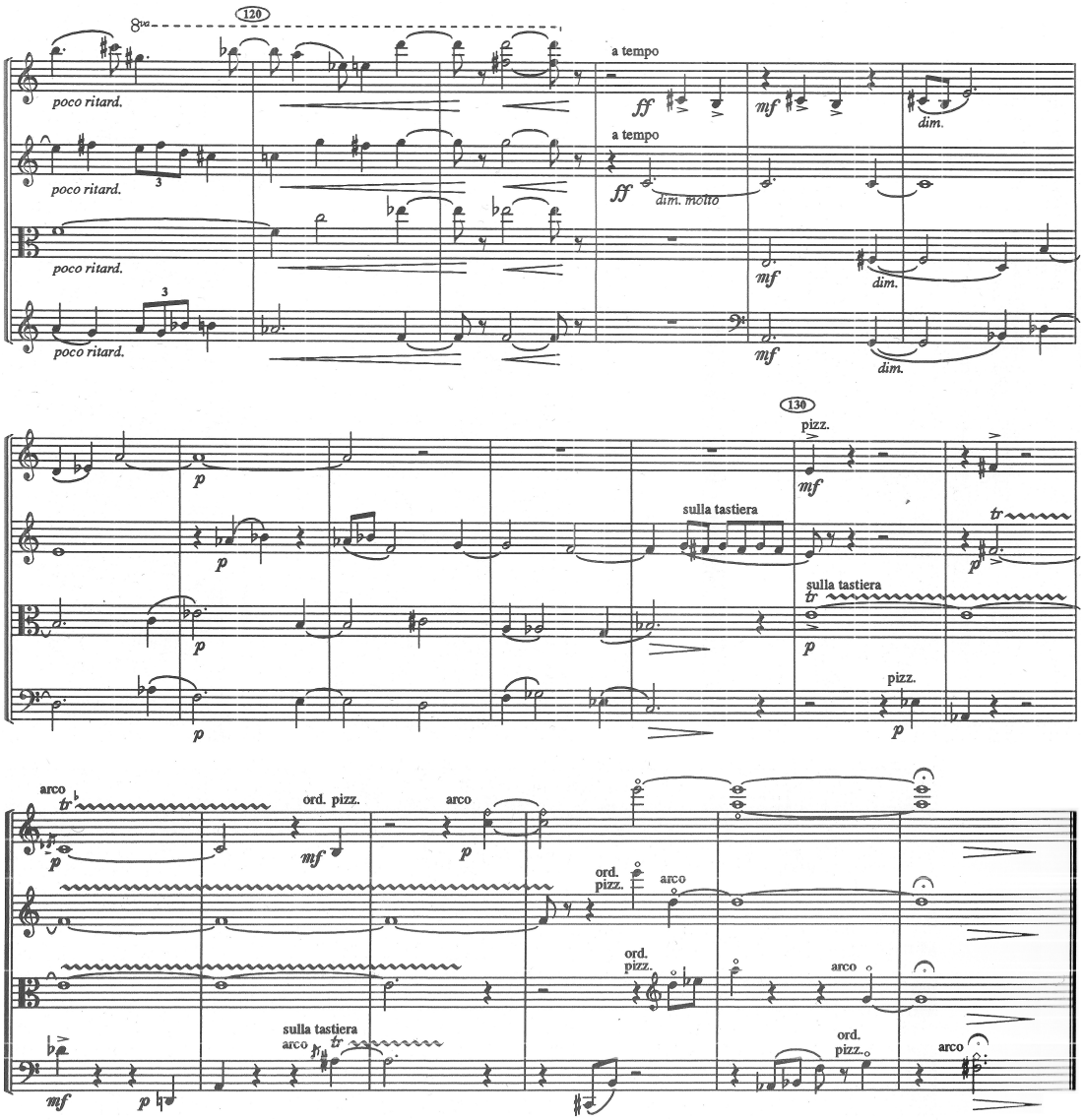
The text used in *Mikrophonie II* is Helmut Heißenbüttel's poem *Einfache grammatische Meditationen* (Simple Grammatical Meditations). Eight quotes of earlier Stockhausen works are heard in moments {2,5,8,11,17,19,20,32} – excerpts from *Gesang der Jünglinge*, *Carré*, and *Momente*. Stockhausen said in an interview with Piergiorgio Odifreddi in 2004 [61] that he used FS in many works, for the "duration of different sections as well as for the density of sound masses." In [56], Kramer discusses the temporal utilization of FS in *Klavierstück IX*, which was begun in 1954 and finished in 1961.

4.2.9 Joonas Kokkonen's *Quartetto II* (1964–1966)

Joonas Kokkonen applied FS to proportion in the first movement in his second string quartet [62], a twelve-tone work. Edward Jurkowski provides a detailed analysis

of the first movement, *Passacaglia*, in [63]. According to Jurkowski, the lengths of the sections defined by the theme of the passacaglia are derived from FS. In the third section in Jurkowski's analysis, for example, he points out how the length of the sections defined by the theme of the passacaglia in measures are $< 3, 5, 8, 13, 8, 5, 3 >$. This indicates the use of reversal without repetition; we also see some use of reversal with repetition in Jurkowski's analysis. While we do not have room to look at a large part of the score, we can at least see the tail end of this section $< 8, 5, 3 >$ in the final bars of the movement shown below, beginning at *a tempo*:

Figure 44: Joonas Kokkonen's *Quartetto II*.

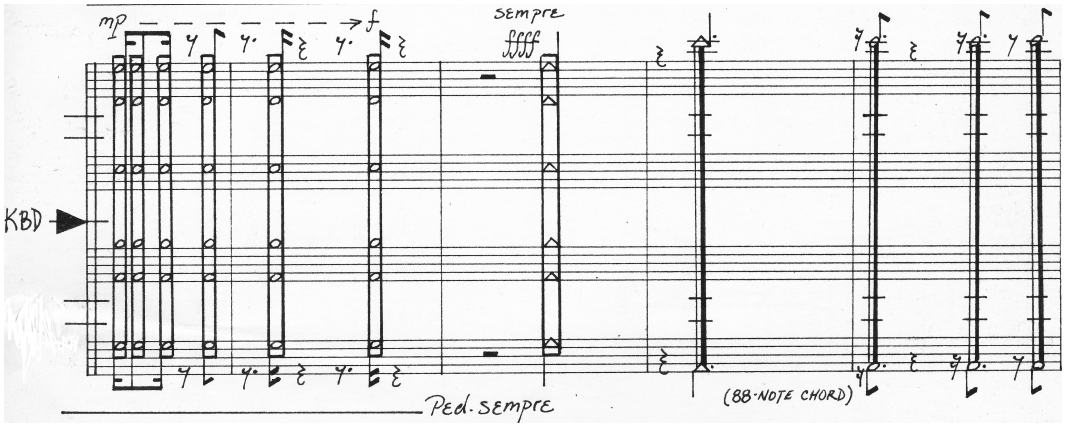


According to my own correspondence with the Finnish composer Mikko Heiniö, who was Kokkonen's student, "[Kokkonen] was very keen on the golden ratio."

4.2.10 Richard Bunger's *Pianography, Fantasy on a Theme by Fibonacci* (1971)

Richard Bunger made use of Fibonacci pulse summation in his work *Pianography, Fantasy on a Theme by Fibonacci* (for monaurally amplified piano with ring modulator and tape recorder) [64]. The composer states that ideally, the speakers of the amplification system surround the audience. The sound of the tape recorder, which is not connected to the amplification system, emanates from the piano directly. The ring modulator is utilized starting at 2 minutes and 7 seconds, set to a constant frequency of 330 Hz. Just before the entrance of the tape loop, near the end of the composition, the pianist plays clusters utilizing a specially constructed bar:

Figure 45: Richard Bunger's *Pianography, Fantasy on a Theme by Fibonacci.*

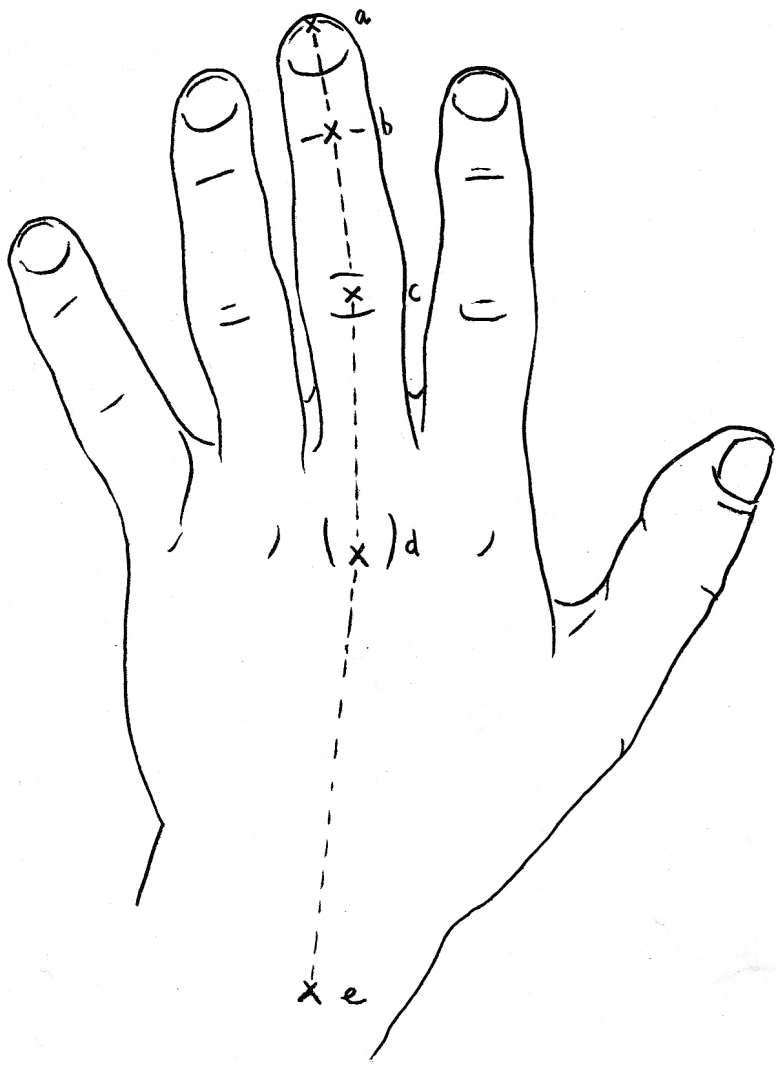


We observe the sequence $< 1, 2, 3, 5, 8, 13, 12, 13, 8, 5, 3, 2, 1 >$ in the number of sixteenth notes each attack is attributed (the last two attacks are not shown above). In using the number 12 in this sequence, which is not a Fibonacci number, perhaps the composer wanted to purposely break the pattern just before the 88-note chord occurs. Similar applications of FS occur throughout. An application of Fibonacci pulse summation utilizing reversal with and without repetition can be observed in measures 9 to 22.

4.2.11 Per Nørgård's *Canon for Organ* (1970–1972)

It was mentioned earlier that a direct application of GR is rare and – according to Xenakis – not possible in instrumental music. However, Per Nørgård applies GR directly to rhythmic values intended to be realized by a performer in his *Canon for Organ* [65], dedicated to Jesper Jørgen Jensen. In the preface to that work, he states that the "Golden Section is known as an essential, irrational division in the shapes of nature and art." Like Schillinger and Xenakis, he demonstrates an interest in the role of GR in biology, including a drawing in the score showing GR in the proportions of a human hand:

Figure 46: Hand with GR in the score of Per Nørgård's *Canon for Organ.*

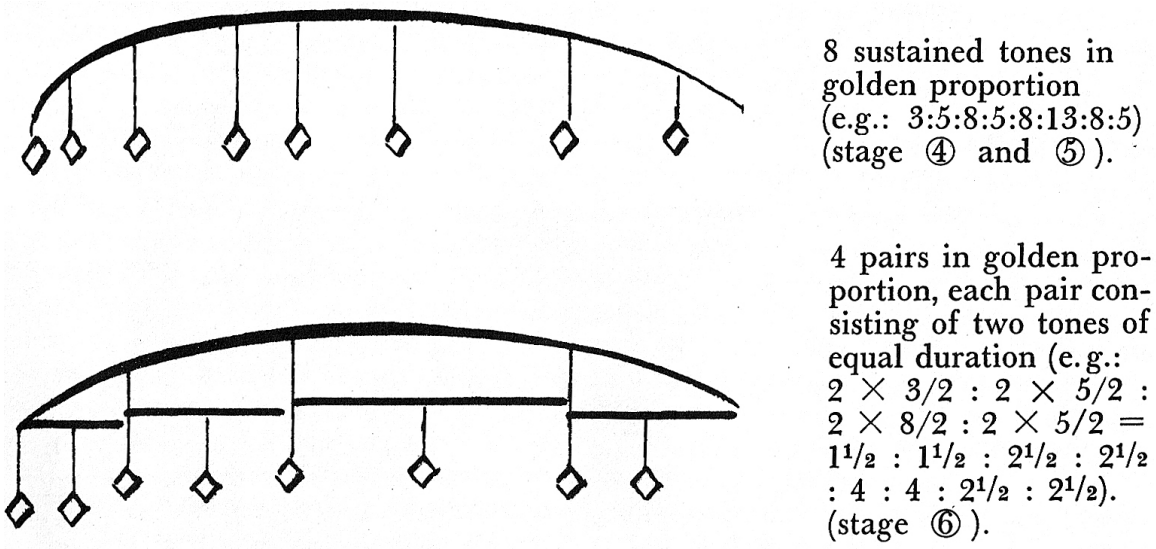


He summarizes the *Canon* as follows:

> 'Canon' presents for the first time The Golden Section transferred to the durations of tones and groups of tones. The Golden Section being an irrational number an adequate notation necessarily must be proportional. In order to make an organic adjustment to the demands of this notation possible, a metric version is published at the same time as the proportional versions. The values of the metric version are necessarily approximate (regarding all 'golden' durations), but the organist is recommended to study, practice, and possibly perform this version as a start. When he has arrived at an organic feeling of the rhythmical proportions, it will be the adequate time to start the practising directly from the proportional version.

Nørgård also gives examples of the types of rhythms that occur and how they are approximated with Fibonacci numbers in his "Commentary on The Proportional Notation of 'Canon'":

Figure 47: Rhythm in Per Nørgård's *Canon for Organ.*



One can observe the use of reversal without repetition in the first example above.

The rhythmic and metric structure of the seven parts of *Canon*, each of which consists of eight lesser parts, is also derived from FS. We observe a proportional application of FS, and reversal with and without repetition in the seven parts:

Figure 48: Proportional application of FS in Per Nørgård's *Canon for Organ.*

The mathematical relation on which the length of the stages is based in the various cycles is

1 : 1 : 1 : 1 : 1 : 1 : 1 : 1 in I,
1 : 2 / 2 : 1 / 1 : 2 / 2 : 1 in II,
2 : 3 : 5 : 3 / 3 : 5 : 3 : 2 in III,
3 : 5 : 8 : 5 : 8 : 13 : 8 : 5 in IV,
3 : 3 / 5 : 5 / 8 : 8 / 5 : 5 in V,
3 : 3 : 3 : 3 / 5 : 5 : 5 : 5 in VI, and
3 : 3 : 3 : 3 : 3 : 3 : 3 : 3
(ɔ: 1 : 1 : 1 : 1 : 1 : 1 : 1 : 1) in VII.

The composer states, however, that various sections within this diagram are repeated and that VII actually has 14 smaller parts.

4.2.12 John Chowning's *Stria* (1977)

John Chowning also applied GR directly in his computer-generated composition *Stria.* A thorough analysis of the composition was carried out by Matteo Meneghini [66].

Chowning, the inventor of FM synthesis, had been trying to find an inharmonic ratio to redefine the concept of the octave. In an interview with Curtis Roads [67], Chowning explains:

> I was looking for an inharmonic ratio such that the components would be powers of some basic ratio. It turns out that the Golden Mean (1.608) [sic] is such a number. If one has a [carrier to modulator] ratio that is 1 to some power of the Golden Mean, then several of the low-order spectral side components are also powers of the Golden Mean.
>
> What I did was draw an analogy between this inharmonic spectrum – including a frequency space where the pseudo-octave is at powers of the Golden Mean – and the harmonic series and tonality, where the low-order components of the harmonic series are also the principal intervals of the tonal system – the octave, the fifth, and so on. I drew this loose analogy and wrote some programs to help me compose, in particular to help me with the sound synthesis. It was not automatic composition by any means, but there were rules for determining the details of the structure, from the microsound level up to the level of a phrase.
>
> In *Stria*, all frequency components are based on powers of the Golden Mean in the c:m ratios. Then I divided up the frequency space so there was some degree of complementarity.

The pseudo-octave in *Stria* was divided into an equal temperament containing nine tones. A central frequency of 1000 Hz was utilized with three pseudo-octaves used above it, and five pseudo-octaves used below it. GR was applied to temporal parameters in *Stria* as well.

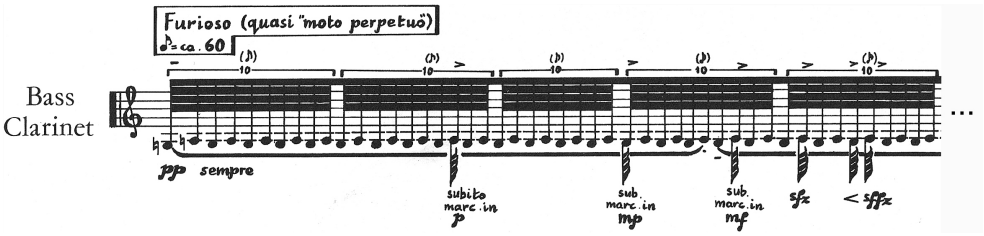
4.2.13 Brian Ferneyhough's *Time and Motion Study I* (1977)

Brian Ferneyhough describes his *Time and Motion Study I* (for solo bass clarinet), dedicated to Harry Sparnaay, as follows [68]:

> *Time and Motion Study I* is built up around a core of two central principles: (1) the opposition and integration of two contrasted types of activity – the one rapid and regular, the other disjointed and gestural, with wide leaps and unpredictable rhythmic patterns; (2) the application, in various roles, of the proportions inherent in the Fibonacci Series (1, 1, 2, 3, 5, 8, 13, …). Right from the outset these worlds are brought into unmediated confrontation.

The presence of the inverse LS is clear in the pattern of accents and changing dynamics at the beginning of the work [69]. There is an accent on the 18th note played. Following this, there is an accent on the 11th note; this is followed by accents on the 7th, 4th notes, etc. This forms an inverse LS $< 18, 11, 7, 4, 3, 1 >$, as seen below:

Figure 49: Brian Ferneyhough's *Time and Motion Study I.*

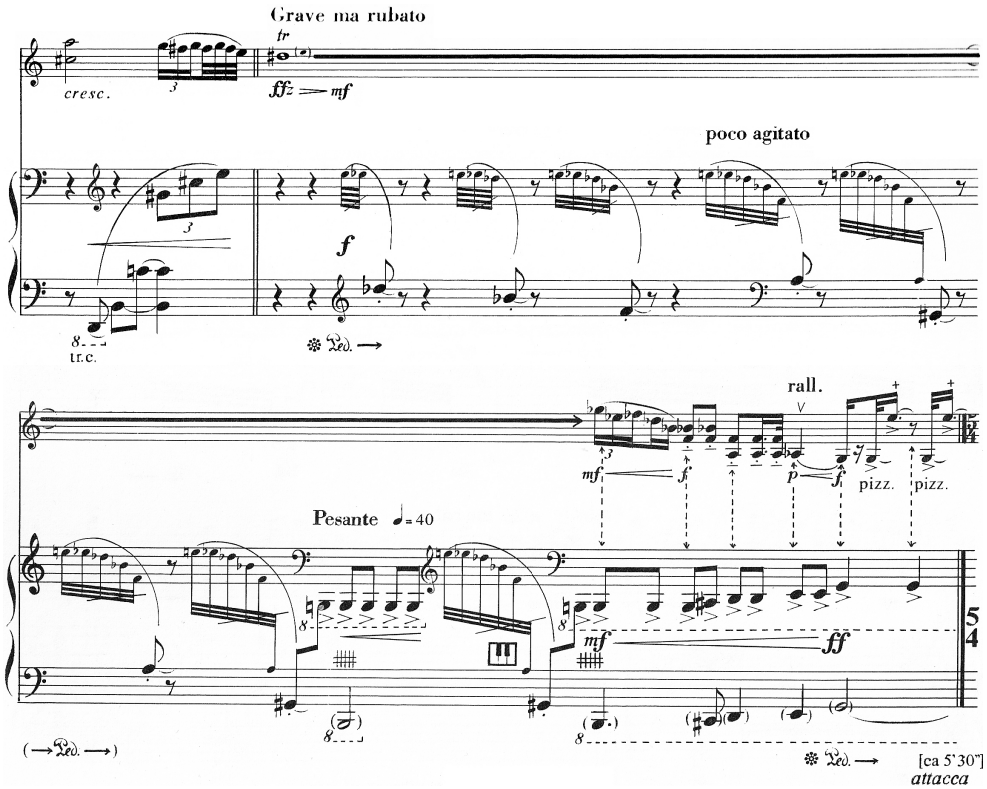


After this, the pattern is broken and accents occur according to the pattern of the Lucas numbers minus one: $< 17, 10, 6, 3, 2 >$. Then he brings in the Lucas numbers $< 3, 4, 7, 11, 18 >$ again in the accent pattern, followed by the Lucas numbers plus one $< 19, 12, 8, 5, 4 >$ and then the Lucas numbers $< 1, 3, 4, 7 >$, after which the pattern breaks up at *subito pesante.*

4.2.14 Mikko Heiniö's *Duo per violino e pianoforte* (1979)

Interval patterns derived from FS similar to those described by Schillinger can be observed in Mikko Heiniö's *Duo per violino e pianoforte* [70]. In the piano part, we observe the following Fibonacci interval sequences in succession: $< 1, 2 >$, $< 1, 2, 3 >$, $< 1, 2, 3, 5 >$, $< 1, 2, 3, 5, 8 >$, $< 1, 2, 3, 5, 8, 13 >$, $< 1, 2, 3, 5, 8 >$, and finally $< 1, 2, 3, 5, 8, 13, 21 >$:

Figure 50: Mikko Heiniö's *Duo per violino e pianoforte.*



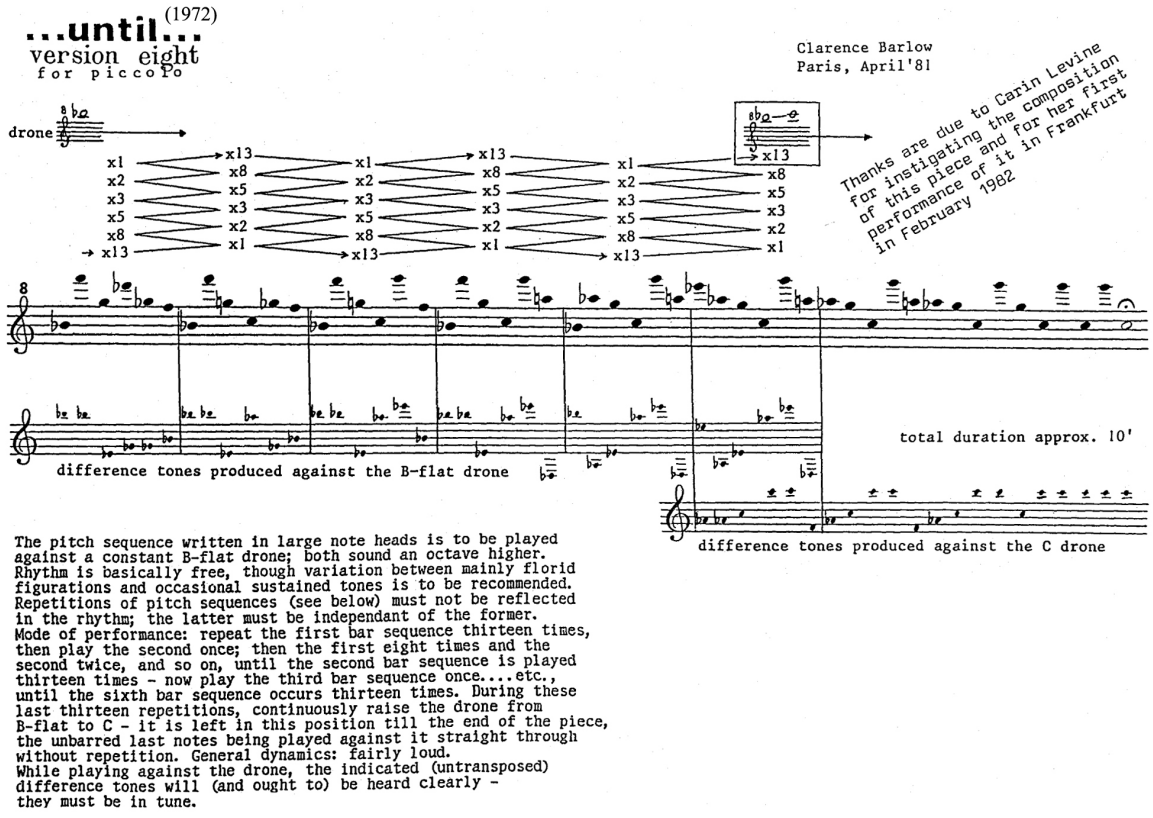
Unordered applications of FS to pitch occur throughout.

According to my own personal correspondence with the composer, he also used FS in his compositions *Tredicia* (1976) and *Notturno di fiordo* (1978). In *Tredicia*, the climax divides the piece into two parts with the ratio 8:5. Metric ratios such as 1:2, 2:3, 3:5, and so on occur throughout the work. The pitch system has nothing to do with FS or GR but is instead based on a twelve-tone chord. In *Notturno di fiordo*, the meter is also derived from FS and all harmonies and melodies are as well.

4.2.15 Clarence Barlow's *... until ... / version eight* (1981)

A unique application of FS occurs in Clarence Barlow's *... until ... / version eight* [71], for piccolo and drone. In this aleatoric work, the number of repetitions of a hexachord are determined by FS. The composer has granted me permission to print the score in its entirety here:

Figure 51: Clarence Barlow's *... until ... / version eight*.

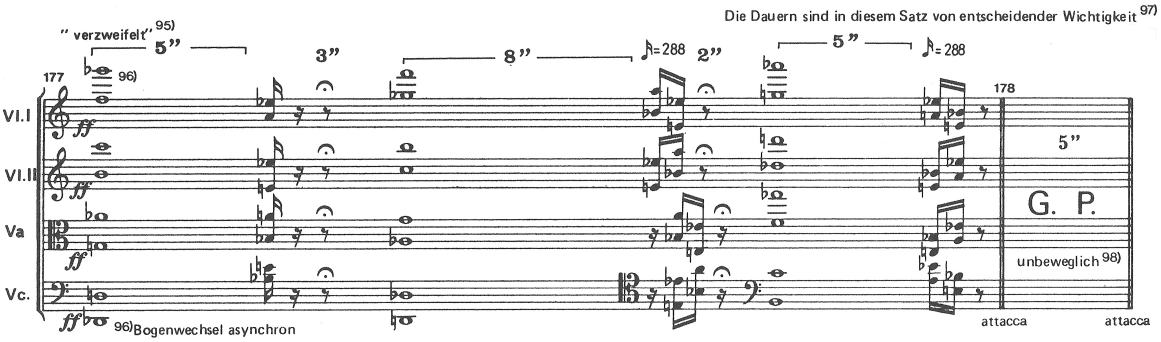


According to personal correspondence, Barlow's interest in FS in this piece has to do with the idea of what he describes as a "quasi-exponential fade" according to FS proportions between the alternating measures, as indicated by the zigzagging arrows in the score; e.g. the first measure slowly "fades" into the second measure according to FS. For Barlow, FS is "a technical tool serving an aesthetic purpose." Barlow also used FS in his *Tschelpowisch* (1968–1971) [72].

4.2.16 Michael Gielen's *Streichquartett* (1983)

In the third movement of his *Streichquartett* [73], Michael Gielen applies FS to temporal proportions. Similar to Stockhausen's application in *Mikrophonie II*, the numbers of FS are not used in order. We can see the use of $< 5, 3, 8, 2, 5, 2, 5 >$ in the first measures of the movement:

Figure 52: Michael Gielen's *Streichquartett*.

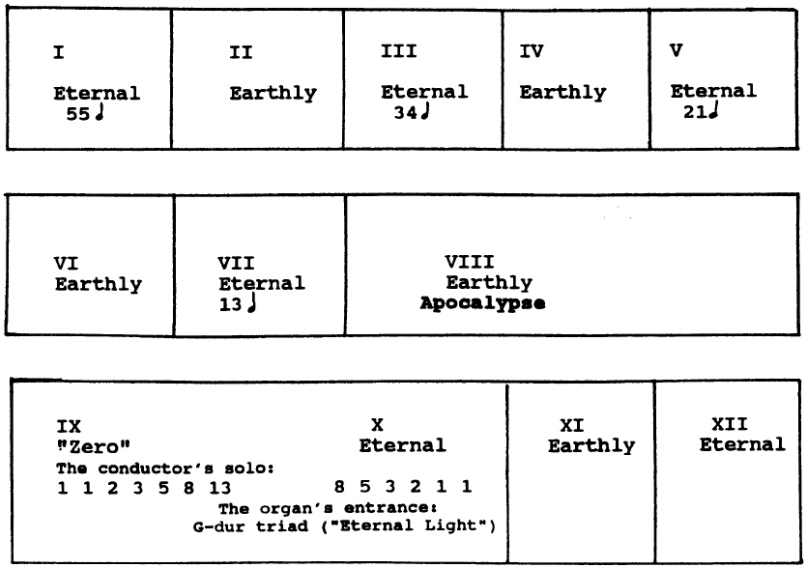


The instructions in German at the top right translate to "The durations are of crucial importance in this movement." Each of the six grand pauses in the work last a Fibonacci number of seconds $< 5, 8, 13, 8, 5, 8 >$. Some proportions utilized in this movement are, however, clearly not Fibonacci-related (e.g. $< 4, 9, 5, 3 >$ in measure 188).

4.2.17 Sofia Gubaidulina's *Stimmen ... Verstummen ...* (1986)

In 1986, Sofia Gubaidulina completed a symphony in 12 movements titled *Stimmen ... Verstummen ...* [74]. According to the composer, all odd movements are organized according to FS, while the even movements are freely composed. Like much of Gubaidulina's work, this work is highly symbolic. Those movements organized using FS are supposed to represent a "harmonic, eternal world," whereas the other movements symbolize a "disharmonic, earthly world." The following graphic, included with an interview with the composer by Vera Lukomsky [75], depicts the twelve movements of the piece:

Figure 53: Movements of Sofia Gubaidulina's *Stimmen … Verstummen …*



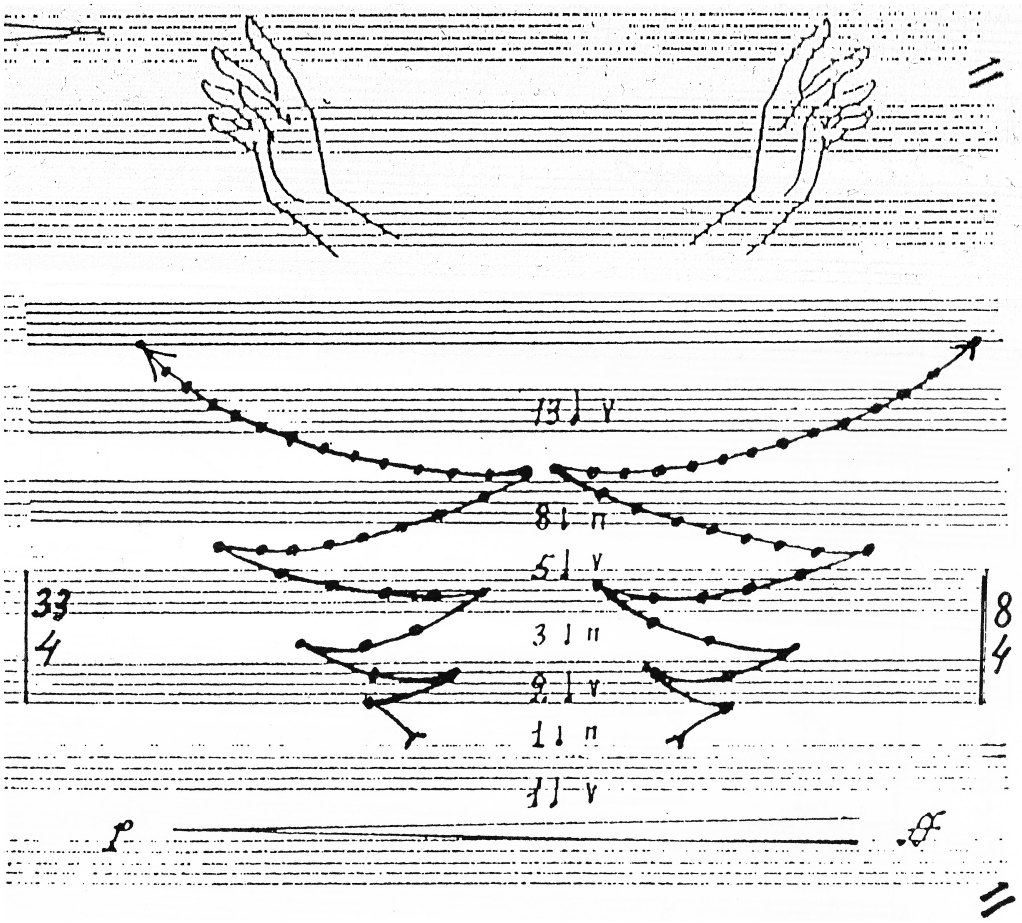
The ninth movement contains a very unique application of FS: a solo for conductor (with occasional percussion or organ accompaniment) that is mostly silent in which the conductor's gestures "erect rhythmic edifices." The composer describes the ninth and tenth movements, shown to be interconnected in the previous graphic, as follows:

> The Ninth [sic] movement is a 'rest': it is a solo for the conductor. It is as if the music had come to 'zero': in the first movement there was [sic] 55 quarters, in the third – 34, in the fifth – 21. [sic] in the seventh – 13, and, finally, in the ninth – zero.
>
> The 'zero' is represented by the conductor's organizing time according to the Fibonacci series: (1, 1, 2, 3, 5, 8, 13). Then he 'holds' 13; at this time the organ enters with a major triad in the high register. For a while, the conductor continues to 'hold' that pyramid of Fibonacci numbers and then starts moving down the Fibonacci series, from 13 to 1.

This indicates the use of reversal without repetition.

An example of one measure from the ninth movement is shown in the following graphic [74]:

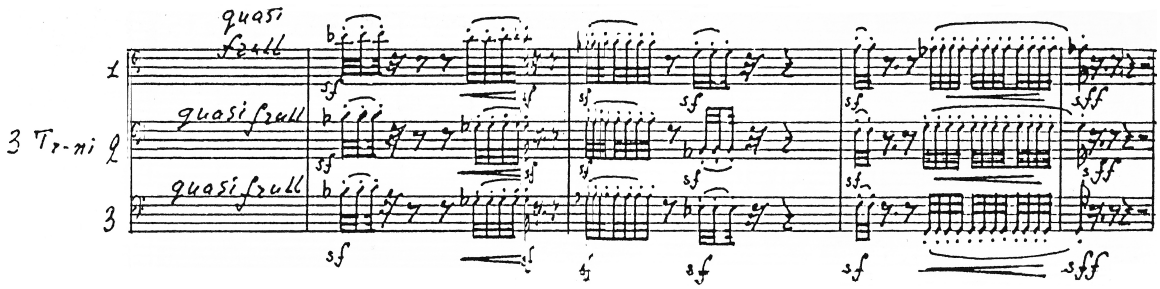
Figure 54: The ninth movement of Sofia Gubaidulina's *Stimmen … Verstummen …*



This gestural application of FS with pulse summation is unique among the pieces analyzed in this section.

In the last movement of the work, FS is applied to the number of attacks in certain gestures – the numbers are nonetheless utilized out of order for the most part. This can be seen in the timpani parts on p. 192 of the score:

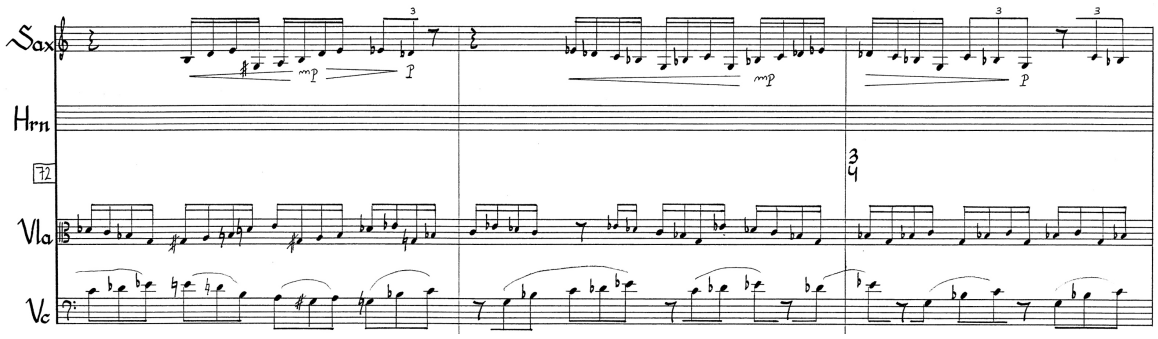
Figure 55: The final movement of Sofia Gubaidulina's *Stimmen … Verstummen …*



4.2.18 Haukur Tómasson's *Octette* (1987)

According to personal correspondence, Haukur Tómasson applied FS and GR to multiple musical parameters in *Octette* (for flute, clarinet, alto saxophone, French horn, viola, violoncello, piano and percussion) [76]. Almost all of the intervals that occur in the piece are derived from FS. Their arrangement seems to be intuitively defined, however, although we see some isolated occurrences of reversal without repetition (e.g. measure 76 in the piano). The use of intervals based on Fibonacci numbers can be seen in measures 72–74, where the only intervals that occur are {1, 2, 3, 5, 8}:

Figure 56: Haukur Tómasson's *Octette*.

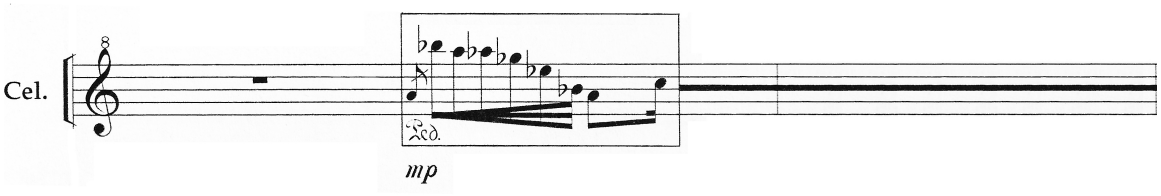


According to the composer, in addition to applying FS to pitch in this work, he also applied FS to various other parameters. The number of players is, for example, a Fibonacci number. FS was also applied to note durations (which seems less apparent), to the lengths of sections, and to meter and the number of iterations of notes (as can be seen at the end). Tómasson thinks of FS as a tool that helps him in compositional organization. In addition, he likes the rhythms and intervals that FS can generate because they approximate GR. He also used FS in his *Eco del passato* (for flute and harpsichord), *Spiral* (for 13 players) and *Annual Ring* (for ensemble).

4.2.19 Harold Blumenfeld's *Diluvial* from *Illuminations: Symphonic Fragments After Rimbaud* (1992)

In the score for Harold Blumenfeld's orchestral composition *Diluvial* [77], based on poems by Arthur Rimbaud published in 1886 and dedicated to Gerhard Samuel, one finds the words "Composed after the Fibonacci Series" below the dedication. No description is given of how FS was applied. The metric and proportional structure of the composition does not come across as particularly rigorous (due to the use of accelerandi, tempo ranges, etc.). One can, however, observe the use of a Fibonacci pitch set starting in the 6th measure of section **P**. The celesta plays the Fibonacci pitch sequence $< 1, 2, 3, 5, 8, 13 >$ embedded in a descending repeated pattern. The pattern is interrupted, however, by the first note and the last two notes (the grace note A and the last A and C in the pattern), which are not part of the Fibonacci pitch sequence:

Figure 57: Harold Blumenfeld's *Diluvial.*

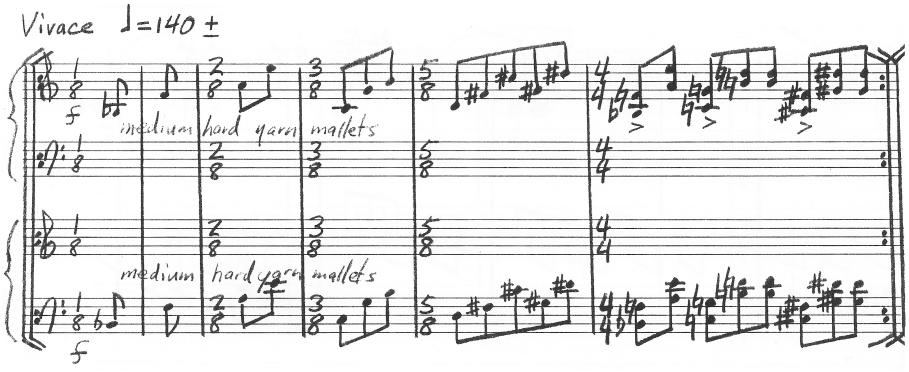


In the two measures before section **O**, we see the descending Fibonacci pitch sequence $< 0, 1, 1, 2, 3 >$ in the horns as well. Most other pitch material in the composition does not appear to be related to FS.

4.2.20 William Hill's *Abstract Relativity* (1999)

In a percussion duet utilizing two marimbas and a vibraphone by William Hill, Fibonacci metric summation is utilized in multiple sections. We can see the use of $< 1, 1, 2, 3, 5, 8 >$ in the marimba parts in the opening bars of the piece:

Figure 58: William Hill's *Abstract Relativity.*



Similar application of FS to meter occurs throughout. We also see the utilization of reversal with repetition in the metric application of FS in the last 16 measures, where the number of eighth notes in each measure follows the sequence $< 1, 1, 2, 3, 5, 5, 3, 2, 1, 1, 1, 1, 2, 3, 5, 1 >$.

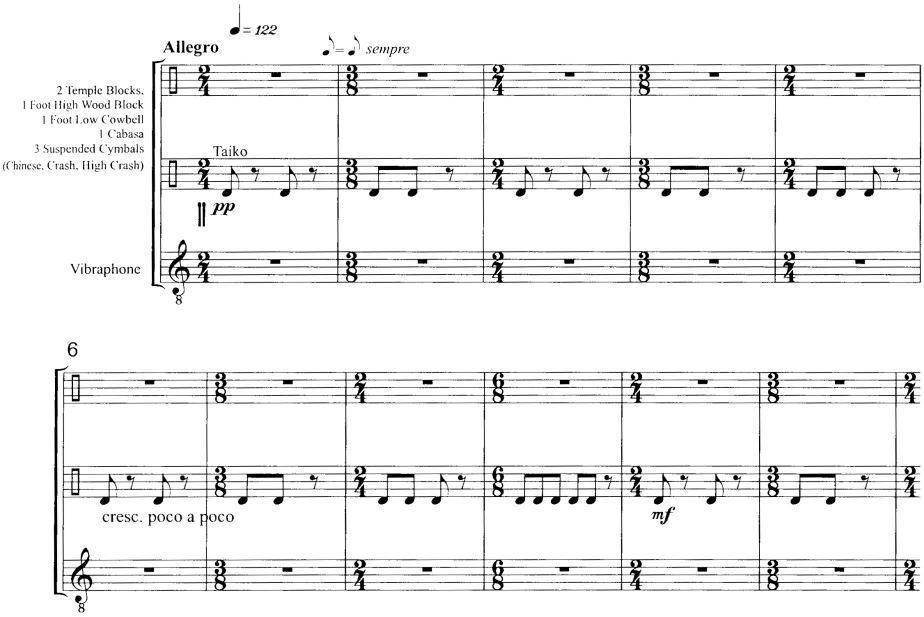
According to the introduction in the score [78], this composition is inspired by the work of Leonardo Fibonacci, Hieronymus Bosch, and Arnold Schoenberg. It integrates twelve-tone rows and juxtaposes "contorted dissonance and lyric consonance much as Bosch placed his visions of heaven and hell side by side."

4.2.21 Ramón García i Soler's *Scherzo con Fibonacci* (2002)

Ramón García i Soler uses FS in the rhythmic pattern in the opening measures of his *Scherzo con Fibonacci* for solo percussion [79]. The first section of the work, the beginning of which is shown below, contains the rhythmic patterns $< 1, 1, 2 >$,

$< 1, 1, 2, 3 >$, $< 1, 1, 2, 3, 5 >$, and $< 1, 1, 2, 3, 5, 8 >$. The numbers in the sequences are separated by eighth note rests:

Figure 59: Ramón García i Soler's *Scherzo con Fibonacci*.

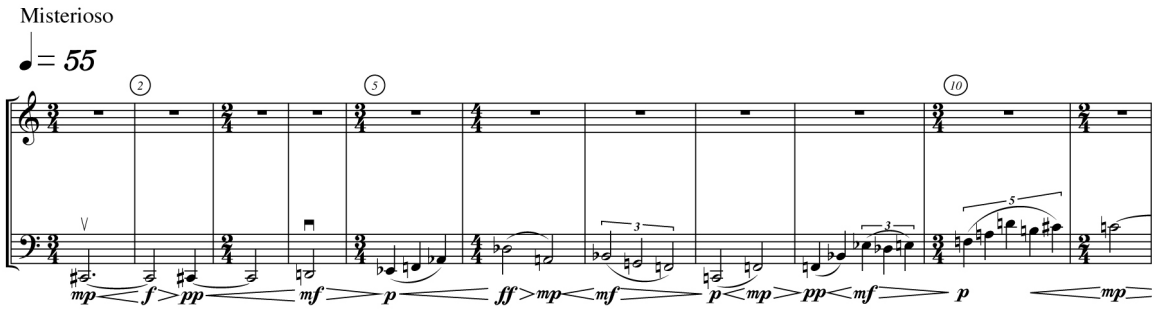


The work subsequently makes use of many other rhythmic patterns that do not appear to be derived from FS.

4.2.22 Kiyomitsu Odai's *Passacaglia di Fibonacci* (2008–2010)

Kiyomitsu Odai's orchestral work *Passacaglia di Fibonacci* is the most thorough application of FS we will encounter in these analyses [80]. In the opening bars of a canon not entirely dissimilar to the first movement of Bartók's *Music for String Instruments, Percussion and Celesta*, the theme consists of one period of an ascending Fibonacci pitch class sequence where the note C = 0, beginning $< 1, 1, 2, 3, 5, 8, 1, 9, 10, 7, 5, 0, 5, \cdots >$:

Figure 60: Kiyomitsu Odai's *Passacaglia di Fibonacci*.



There are 610 quarter notes in the work, and strict tempo is kept throughout. Major sections consist of Fibonacci numbers of quarter notes. A climax is reached on the 378th quarter, approximating GR with great precision (measure 124). Meters containing exclusively Fibonacci numbers of quarter notes occur throughout (instead of using an 8/4 meter the composer uses two 4/4 meters). In the metric structure, we see heavy use of reversal with and without repetition, and some unordered application of FS as well. Starting in measure 135, the orchestra "pronounces" the syllables "Fi," "bo," "na," "cci" through spectral representation: the consonances are produced by short grace notes on the beat, and the vowels by longer notes.

According to personal correspondence, Odai compares FS to a landscape. He has used FS and GR in nearly a dozen works – though recently less frequently – and is fascinated by the principle of infinite convergence and "the balance between symmetry and asymmetry."

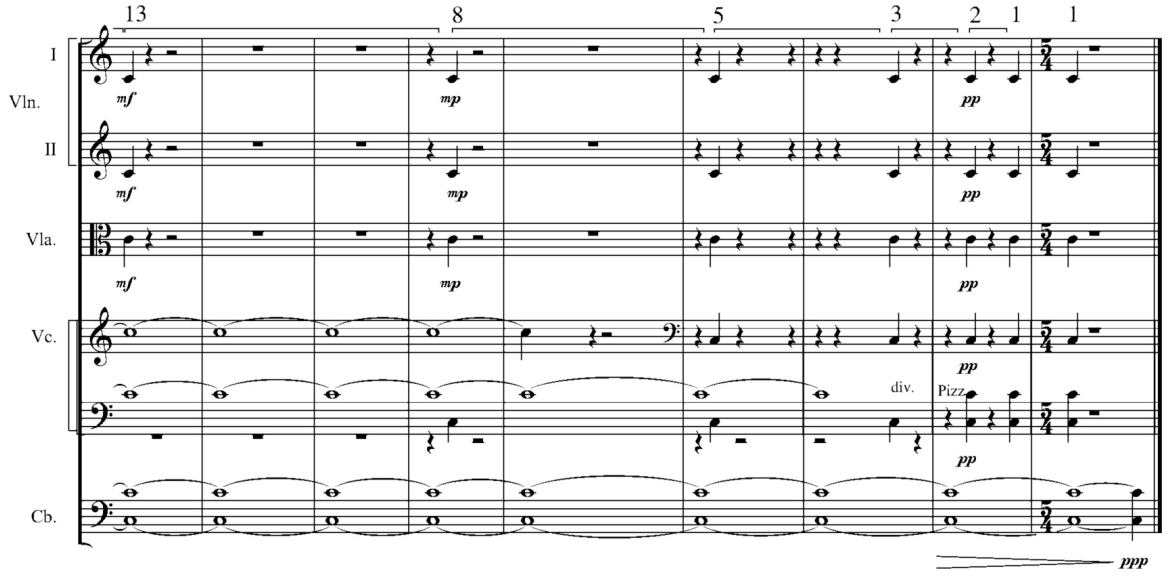
4.2.23 Da Jeon Choi's *Dream of a Thousand Keys: A Concerto for Piano and Orchestra* (2011)

Da Jeon Choi describes the application of FS in her piano concerto *Dream of a Thousand Keys* in her dissertation [81] as follows:

> The Fibonacci proportions not only determine the form of the entire work but also the structure, tempo, meter, duration, rhythmic patterns, phrase length, and pitch collections.

In the second movement of this work, dedicated to her father Nak Joong, we observe the use of inverse Fibonacci pulse summation in the final measures 268 to 276:

Figure 61: Da Jeon Choi's *Dream of a Thousand Keys*.



The pitch material in the work is based on a twelve-tone row derived from the Fibonacci modulo 12 pitch class sequence. The pitches of the twelve-tone row used are shown below in their correct order in bold, showing their derivation from a Fibonacci pitch class sequence – the final pitch **6**, which never actually occurs in the Fibonacci sequence modulo 12, is appended to the end to complete the row:

0 1 1 **2 3 5 8** 1 **9 10 7** 5 0 5 5 10 3 1 **4** 5 9 2 **11** 1

According to personal correspondence, Choi is very interested in mathematical concepts and considers them one of the greatest tools in constructing her compositions. There is an orchestral version of the work entitled *From Dawn to Sunset* in which the same application of FS occurs. In addition to the use of FS, the work also integrates elements of traditional Korean music, jazz and aleatorics. This is the only work as of yet in which Choi has utilized FS.

4.3 Overview of Analyses

As can be seen, GFS and GR have been utilized in many different ways by composers, and certain trends have emerged. In order to gain some oversight of the applications we have seen in these analyses, we will construct a table that indicates the types of applications we have observed. Only those applications that we have noted in the pieces analyzed will be indicated in the table.

Because the applications we have seen are so diverse and numerous, it will be necessary to group certain applications together. Use of the inverse GFS and GFS are not differentiated in this table; one can, however, infer that both were used if the utilization of reversal is indicated. Although metric and rhythmic applications can also be considered smaller scale proportional applications, the category "Proportional Application of GFS" in the table refers to larger scale temporal structure, as described earlier in 4.1.4. Unordered pitch and temporal/proportional applications of GFS (where the numbers of GFS appear out of order) are indicated in the categories "Other Pitch Application of GFS" and "Other Temporal Application of GFS." Only those applications in which at least three numbers of a GFS $< G_n, G_{n+1}, G_{n+2} >$ are utilized in order are considered to be ordered applications, unless such an ordering appears to be coincidental and isolated.

Figure 62: Overview of analyses.

composer (year)	Pitch Sequences Based on GFS	Pitch Class Sequences Based on GFS	Other Pitch Application of GFS	Attack Summation Based on GFS	Pulse Summation Based on GFS	Metric Summation Based on GFS	Proportional Application of GFS	Other Temporal Application of GFS	Serial Application of GFS	Reversal With Repetition	Reversal Without Repetition	Direct Application of GR
Bartók (1936)				×							×	
Schillinger (1941)	×	×										
Blacher (1950)						×					×	
Xenakis (1952)					×	×		×			×	
Nono (1956)									×	×		
Amenábar (1957)							×	×		×	×	
Krenek (1964)								×				
Stockhausen (1965)								×				
Kokkonen (1966)							×			×	×	
Bunger (1971)					×					×	×	
Nørgård (1972)							×	×		×	×	×
Chowning (1977)												×
Ferneyhough (1977)								×				
Heiniö (1979)	×		×									
Barlow (1981)								×				
Gielen (1983)								×				
Gubaidulina (1986)					×			×			×	
Tómasson (1987)			×					×			×	
Blumenfeld (1992)	×											
Hill (1999)						×				×		
García i Soler (2002)								×				
Odai (2010)	×	×				×	×	×		×	×	
Choi (2011)			×		×							

As can be seen, temporal applications of GFS are most common, and have taken a variety of contrasting forms. The application of GFS to pitch and the direct application of GR, on the other hand, are relatively uncommon. Pulse summation appears to be more common than attack summation. We only saw a single, cryptic, serial application of FS. Reversal was utilized in over half of the compositions analyzed, and reversal without repetition appears to be more common than with. We observed no applications of GFS to dynamics or spatial location.

4.4 Obfuscation of GFS and GR in the Works of Twentieth and Twenty-First Century Composers

The composers discussed in this section had various goals in their application of GFS and GR, and it would appear that none had the chief purpose of creating a direct sonification of the mathematical objects used. In all cases, the composers utilized GFS and GR combined with numerous other highly developed compositional techniques that are entirely unrelated. GFS and GR seem to have been used as a means to a certain end, which is different for each composer. While the results have proven to be interesting and unique for each composer, this combination of techniques obfuscates the presence of GFS and GR in the structure of their music – in some cases so greatly it renders the utilization of GFS or GR practically undetectable. This makes it difficult to discern whether GFS and GR might contribute to making the music more aesthetically pleasing and to what degree. The more arbitrary the choices introduced into a composition unrelated to the mathematics, the less direct the relationship between the music and the mathematical objects utilized. However, because it was not these composers' intention to create pure sonifications of GFS and GR, we will move on and research for ourselves how one might go about preserving the relationship between the music and the mathematical objects utilized in composition. In doing so, we will forge the foundation of a unique musical style that might be of potential use in determining the aesthetic value of GFS and GR in music.

5 Criteria for a Style of Music Characterized by Mathematical Objects Fundamentally Related to FS

These criteria were originally conceived in order to research the direct sonification of mathematical objects related to FS in the greatest depth possible, driven out of curiosity and a desire to discover interesting and rich new musical structures that would not be possible to create and experience through other means. Direct sonification, or pure sonification, is a technique in which one attempts to create a sonic graph that represents as accurate a likeness of the data or mathematical object sonified as is possible. Although we will encounter other types of mathematical objects as well, number sequences – functions of GFS and GR – will represent our primary source of material for sonification. The criteria therefore focus on the sonification of number sequences; the principles of stringency they exemplify, however, could be applied to the sonification of other types of mathematical objects as well, as we will see in the section that follows.

The primary goal of the criteria is to ensure the clarity of the presentation and comprehensibility of the mathematical sequences sonified and, for research purposes, to mitigate combinatorial and associative influences. Above all, we seek to illustrate and bring out structural properties of the mathematical objects in sonification – and we seek to avoid obscuring such.

These criteria do not guarantee a good sonification in themselves: A good practical understanding of psychoacoustics, a sense of good judgment and insight into the nature of the mathematical objects will be necessary in order to create compelling sonifications that are both accurate and aesthetically pleasing. The criteria are also not an attempt at a complete description of the style, nor are they set in stone. More than anything else, the criteria represent an ideal and a certain mode of thinking that places the mathematical objects to be sonified at the core of the compositional process. They are formulated to be general enough to provide for great potential variation, yet strict enough to guard against much of the obfuscation that can occur in sonification.

5.1 Criteria for the Sonification of a Single Integer Sequence or a Single Sequence of Sets of Integers

The criteria that follow apply to the sonification of single integer sequences as well as single sequences of sets of integers. Because each criterion has an explanation, example and rationale, it will be helpful to introduce two of the sequences that are used in the examples beforehand.

A signature sequence is defined as follows. If R is a positive irrational number and we arrange the set of all numbers $i + jR$ in order of increasing value, where i and j are positive integers, $i_1 + j_1R$, $i_2 + j_2R$, $i_3 + j_3R$, ..., then i_1, i_2, i_3, ... is the signature sequence of R [82].

Zeckendorf's theorem [3, 83, 84] guarantees $n \in \mathbb{N}$, where $n > 0$, can be represented as

$$Z(n) = \sum_{j=2}^{M} c_j F_j,$$

where $c_j \in \{0, 1\}$ and $c_j c_{j+1} = 0$. In other words, every positive integer can be represented as a sum of non-consecutive Fibonacci numbers:

$$Z(17) = 1F_7 + 0F_6 + 0F_5 + 1F_4 + 0F_3 + 1F_2.$$

These sets of Fibonacci numbers summing to n can be utilized as a sequence (see A007895 in [2]):

$$< \{1\}, \{2\}, \{3\}, \{3, 1\}, \{5\}, \{5, 1\}, \{5, 2\}, \{8\}, \cdots > .$$

One can sonify the indices of these Fibonacci numbers as well:

$$< \{2\}, \{3\}, \{4\}, \{4, 2\}, \{5\}, \{5, 2\}, \{5, 3\}, \{6\}, \cdots > .$$

Zeckendorf representations and their generalizations will be described in greater detail later.

We will begin by introducing some general criteria and then move on to those that only apply when certain *dynamic parameters* are utilized. Dynamic parameters are those musical parameters that the sequence controls – those that change according to the values of the sequence. This can include parameters such as frequency, level of loudness, spatial location, etc. *Static parameters* are, in contrast, those musical parameters that do not change according to the sequence and remain the same throughout the sonification. This can include many things, such as the number of audio channels used, or the duration attributed to each integer, etc. Dynamic or static parameters can be *individual* (applying to a single sequence) or *global* (applying to all sequences sonified). Under certain circumstances to be detailed, they can be applied to groups of sequences as well.

In the following, an integer sequence will be referred to as a_n, and a sequence of sets of integers as A_n. Min of a_n and max of a_n, or min or max of A_n, will be used to refer to the smallest and largest integers that occur in the sequence or portion of a sequence sonified. Mapping of integers to a dynamic parameter in sonification is indicated with an arrow $a_n \rightarrow$ (dynamic parameter value).

5.1.1 Criterion: Functions of GFS or GR

Explanation: In order to be acceptable for sonification, a sequence must be a function of either GFS or GR.

Example: A Beatty sequence $a_n = \lfloor \varphi * n \rfloor$ would be acceptable for sonification.

Rationale: Our purpose is to explore all types of sequences and mathematical objects related to GFS and GR, driven by curiosity and a desire to discover interesting and rich new musical structures. This focus on mathematical objects related to GFS and

GR lends a great degree of cohesion to the style as a whole that would not be possible if we were to allow sonifications of other types of unrelated mathematical objects as well.

5.1.2 Criterion: Offset

Explanation: A sonification must either begin with a_0 or a_1 (or A_0 or A_1). A composer will have to use a sense of good judgment and understanding of the sequence in determining which is most appropriate for a given sequence.

Example: When using a sequence of sets based on Zeckendorf representations, we begin with A_1, because 0 cannot be represented as a sum of members of the sequence $< 1, 2, 3, 5, 8, 13, \cdots >$.

Rationale: We do not want to leave any information off in the beginning or mislead the listener into thinking the sequence begins in a place it does not.

5.1.3 Criterion: Point of Termination

Explanation: The index of the last member of a sequence used in a sonification is called the point of termination. A sonification should end at a mathematically significant point in the sequence, or at a point that highlights some type of symmetry. Due to the fact that there are infinitely many sequences, it will not be possible to give a general rule regarding this. Often, if a sequence is a function of FS, the point of termination will be of the form F_n or F_{n-1}, we end the composition at a Fibonacci index or a Fibonacci index minus one; similarly, if a sequence is a function of LS, the point of termination will often be of the form L_n or L_{n-1}. If these points seem less mathematically relevant than other points, then another point of termination might be chosen. Sometimes, in an integer sequence, the point of termination is the highest or lowest member that has yet occurred. In a sequence of sets, the point of termination might be the largest set that has yet occurred. In installations, however, it is possible to sonify a sequence without a predetermined point of termination, seeing that the duration of the composition can extend beyond the point where listeners are present. Purely conceptual or aleatoric compositions do not require a point of termination.

Example: Signature sequences (e.g. A084531 in [2]) represent a difficult case, due to the uniformity in the distribution of the integers in the sequence. For signature sequences, we end on either the highest or lowest integer that has yet occurred.

Rationale: If we want to preserve the symmetry of certain sequences, and create aesthetically appealing sonifications as well, then we cannot end at random points in the sequence. In practice, the sonification of a sequence has to end at some point, whether the listener exits the room in an installation or a point of termination is predetermined.

5.1.4 Criterion: Sonification in Order

Explanation: A sequence must always be sonified in order temporally, with no members omitted until the point of termination, e.g. $< a_1, a_2, a_3, a_4, a_5, \cdots >$.

Example: When using a sequence based on sets of Zeckendorf representations, the sequence of sets would be sonified in order, e.g. $< \{1\}, \{2\}, \{3\}, \{3, 1\}, \{5\}, \cdots >$.

Rationale: If the sequence is sonified out of order or members are omitted, then we are sonifying a different sequence.

5.1.5 Criterion: Duration of a Single Member of the Sequence

Explanation: If duration is a static parameter, then a single durational value is attributed to each member of the sequence that remains unchanged throughout. When sonifying A_n, all of the integers contained in a set are sonified simultaneously as a chord. Only when sonifying a single sequence containing only positive integers can duration be used as a dynamic parameter. In the case of an integer sequence, one chooses a unit duration d and the duration of an integer is determined by $d * a_n$. When sonifying A_n with duration as a dynamic parameter, each integer in a set z is attacked simultaneously but receives its own duration $d * z$, making the full duration of the chord determined by max of A_n.

Example: We could have each member of a_n or A_n last 0.05 seconds. Or, if we use a single integer sequence containing only positive integers, we could choose to have the integer 1 last 0.01 seconds, 2 last 0.02 seconds, 3 last 0.03 seconds, etc. The Zeckendorf representations $< \{1\}, \{2\}, \{3\}, \{3, 1\}, \{5\}, \cdots >$ could be sonified consecutively (sets are never allowed to overlap) with the durations – given in seconds – $< \{.01\}, \{.02\}, \{.03\}, \{.03, .01\}, \{.05\}, \cdots >$. Each integer in a set would be attacked simultaneously.

Rationale: Any other option introduces arbitrary choices into the sonification, obfuscating the sequence being sonified. If duration is a dynamic parameter, then the sequence must contain only positive integers because negative integers cannot be represented in time – nor can the integer 0 – without obfuscation. Duration is seldom utilized as a dynamic parameter, primarily because the overall structure of the sequence tends to become less comprehensible psychoacoustically, especially in slower sonifications.

5.1.6 Criterion: 0s Silent

Explanation: 0s can be sonified as silence in the sonification of a_n, but only if the integer 0 represents some type of void. An understanding of the sequence and a sense of good judgment will have to be employed in discerning which sequences can be sonified with silent 0s. 0s cannot be silent in sonifications of A_n.

Example: In a sequence that counts the number of possible representations of n as a sum of distinct numbers of the Lucas sequence beginning $< 1, 3, 4, 7, 11 >$ (for more information, see A003263 in [2] and section 6.9 of this dissertation), the integer 2 cannot be represented as a sum of integers from this sequence, so $a_2 = 0$. This

0 can be represented as silence because there are no possible representations of that integer.

Rationale: A 0 can be effectively represented by silence for integer sequences, providing rhythmic variation where it represents some type of void, and can help to highlight the properties of the sequence as such. If we allowed silent 0s in sets, then we could cause obfuscation in that one would not know if a 0 was present in a chord.

5.1.7 Criterion: Empty Sets Silent

Explanation: Empty sets in A_n must be sonified as silence.

Example: If A_n is $< \{1\}, \{\}, \{3\}, \{4\}, \{1, 4\}, \{\}, \{7\}, \{1, 7\}, \cdots >$, then the empty sets must be sonified as silence.

Rationale: Silence is the best possible musical representation of an empty set.

5.1.8 Criterion: All Parameters Dynamic or Static

Explanation: A parameter must be either dynamic or static, i.e. it must change according to the sequence or remain unchanged throughout.

Example: Spatial location must either remain the same throughout or be controlled by the sequence; no other option is acceptable.

Rationale: If a musical parameter were to be anything else besides static or dynamic, then we would be introducing additional and unnecessary arbitrary elements into the sonification.

5.1.9 Criterion: What Dynamic Parameters are Permissible

Explanation: In order for a musical parameter to be acceptable for use as a dynamic parameter, there must be some perceptually linear gradations possible using it.

Example: Loudness could be used because it has perceptually linear gradations such as *pp*, *p*, *mp*, *mf*, *f*, *ff*, etc.

Rationale: Only dynamic parameters that have such perceptually linear gradations are effective at conveying a mathematical sequence in sound. If linear gradations are not possible, then mapping of the integers cannot occur.

5.1.10 Criterion: At Least One Dynamic Parameter

Explanation: A sonification must make use of at least one dynamic parameter.

Example: Frequency could be used as a dynamic parameter.

Rationale: If there is no dynamic parameter, then sonification cannot take place.

5.1.11 Criterion: Perceptually Linear Mapping of Dynamic Parameters

Explanation: When a dynamic parameter is chosen for a sonification, one assigns a specific value of that dynamic parameter to min of a_n (or A_n), and a different

value to max of a_n (or A_n). The integers that fall in between the lowest and highest members must be attributed perceptually linear gradations falling in between the lowest or highest values assigned of that dynamic parameter. A single exception is made for the harmonic series, explained in the tuning criterion to follow. The values the integers are mapped to cannot be changed in the course of sonification in any case.

Example: If loudness is a dynamic parameter, and min of $a_n = 0$ and max of $a_n = 5$, then we would assign the value ***pp*** to min of a_n and ***ff*** to max of a_n. The integers would then be mapped to loudness linearly as follows: $0 \rightarrow$ ***pp***, $1 \rightarrow$ ***p***, $2 \rightarrow$ ***mp***, $3 \rightarrow$ ***mf***, $4 \rightarrow$ ***f***, $5 \rightarrow$ ***ff***.

Rationale: In order to preserve the relationship between the sequence and its sonification, we need to use a perceptually linear mapping. Every time a particular integer occurs, it should sound the same.

5.1.12 Criterion: Tuning

Explanation: If frequency is a dynamic parameter, then two types of tunings may be utilized. The first type of tuning, which is equally tempered, has a unit interval equal to $\varphi^t + 1$, where $t \in \mathbb{Z}^+$. In this case, one can either have *ascending* or *descending frequency orientation*. In ascending orientation, higher integers are represented by higher frequencies; in descending orientation, higher integers are represented by lower frequencies. The second acceptable type of tuning is the harmonic series. In this case, there is a stipulation that the sequence must contain only non-negative integers. The integers are then mapped directly to frequency $\omega * a_n$, where ω is a predetermined fundamental frequency. 0 is then silent by default. Only a single tuning may be used in a sonification.

Example: We could use φ^7+1 as a unit interval in a tuning, and assign the frequency 850 Hz to the integer 1. If our orientation was ascending, then we could map the integers as follows: $1 \rightarrow$ 850.0 Hz, $2 \rightarrow$ 879.3 Hz, $3 \rightarrow$ 909.6 Hz, etc. Or, if utilizing the harmonic series with a sequence containing only non-negative integers, we could select $\omega = 100$ Hz, and then the frequencies of the integers are mapped as follows: $1 \rightarrow$ 100 Hz, $2 \rightarrow$ 200 Hz, $3 \rightarrow$ 300 Hz, etc. Once the tuning is chosen for a sonification, it cannot be replaced with another.

Rationale: We first desire a system of equal temperaments containing no octaves (in order to avoid pitch class equivalence, which can result in ambiguities) that can be derived from FS and GR:

$$\lim_{n\to\infty} \frac{F_n + F_{n-t}}{F_n} = \varphi^t + 1 \,.$$

As mentioned in the linear mapping of dynamic parameters criterion, a single exception to that criterion is made for the harmonic series. The harmonic series is not necessarily perceived as a linear mapping, but it is allowed because of the easy comprehensibility and identification of the integers in that tuning, and the direct mapping to frequency $\omega * a_n$. Negative integers cannot be represented using the harmonic series

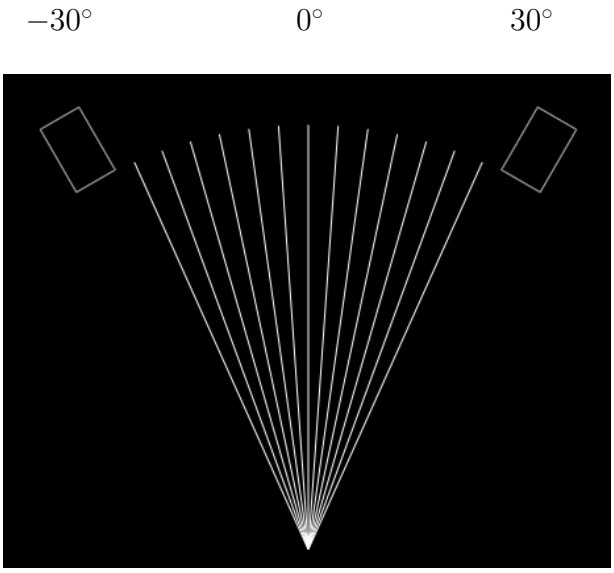
in any psychoacoustically meaningful way, therefore the sequence must contain only non-negative integers. Using more than one tuning in a sonification would violate the linear mapping of dynamic parameters criterion.

5.1.13 Criterion: Bilateral Symmetry in Spatialization

Explanation: If spatial location or simulated spatial location is a dynamic parameter, then the integers must be equidistantly arranged (with the listener facing forward), either in a line or on a circle with the listener in the center, with a radius equal to the distance of the listener to the speaker. If 0° represents the location directly in front of the listener, and 30° represents a full stereo pan to the right, and -30° degrees represents a full stereo pan to the left, then the absolute value for the angles attributed to min of a_n or A_n and max of a_n or A_n must be equal. The integers in between must be perfectly evenly distributed according to the linear mapping of dynamic parameters criterion.

Example: If min of $A_n = 1$ and max of $A_n = 13$, then one might arrange the location of the integers 1 through 13 as in the following image, where the boxes represent two loudspeakers and the line segments represent the path from the simulated source location of the integers 1 through 13 and their path towards the listener seated where the line segments converge:

Figure 63: Bilateral symmetry in simulated sound sources.



In this case, min of A_n is attributed the angle -24° and max of A_n is attributed 24° – a *clockwise spatial orientation.*[10] The integers 2 through 12 fall in between according to the linear mapping of dynamic parameters criterion.

[10]Location is given in degrees, with negative numbers indicating a sound source emanating from the left and positive those from the right.

Rationale: The clearest and most beautiful presentation is that in which bilateral symmetry is utilized.

5.1.14 Criterion: Temporal Accuracy

Explanation: If human performers are utilized, then some means of electronic synchronization (such as a click track) must be used.

Example: If the duration of a single member of the sequence is defined to be 0.5 seconds, then a click track could be created that the performers use.

Rationale: We want to ensure the exactness of the temporal proportions in a sonification so as not to obfuscate the mathematical object being sonified.

5.2 Criteria for the Simultaneous Sonification of Multiple Integer Sequences or Multiple Sequences of Sets of Integers

All of the criteria that preceded also apply in the sonification of multiple sequences simultaneously. In order to maintain comprehensibility in the sonification of multiple integer sequences, it will be necessary to introduce additional criteria.

5.2.1 Criterion: Relatedness

Explanation: The sequences sonified must be related in some sense mathematically or the composer must be seeking to highlight some salient or easily recognizable relationship between them; one cannot simply choose any number of random sequences to sonify simultaneously. For this criterion, a composer will have to rely on a sense of good judgment and an understanding of the sequences in choosing which sequences can be sonified together and which cannot.

Example: LS $< 2, 1, 3, 4, 7, 11, 18, \cdots >$ and FS $< 0, 1, 1, 2, 3, 5, 8, \cdots >$ are clearly related, because they have the same defining relation, and could thus be sonified simultaneously. FS and the decimal expansion of GR $< 1, 6, 1, 8, 0, 3, \cdots >$ (A001622 in [2]), however, would not be closely enough related for simultaneous sonification.

Rationale: If we were to compare integer values that refer to things that are almost entirely unrelated, then the result would be confusion and lack of clarity in sonification. We seek a certain degree of harmony when sonifying multiple sequences.

5.2.2 Criterion: Point or Points of Termination

Explanation: The number of members of the sequences sonified must be either the same for all sequences, or the composer must be seeking to highlight some relationship by choosing different points of termination. Generally, the composer seeks to end at some mathematically significant point, or a point that highlights some type of symmetry, which is often but not always the same for all sequences sonified. This is again a case in which the composer will have to use a sense of good judgment and an understanding of the nature of the sequence, seeing that there are infinitely many

integer sequences and it would therefore be impossible to give strict rules regarding this.

Example: We could sonify two Beatty sequences simultaneously, $a_n = \lfloor \Phi * n \rfloor$ and $b_n = \lfloor \Phi^2 * n \rfloor$ (A000201 and A001950 in [2]), and use $F_9 = 34$ members of each sequence.

Rationale: Ending at random points in the sequence could be distracting and obscure the relationship between the sequences.

5.2.3 Criterion: Shared Static and Dynamic Parameters

Explanation: All sequences sonified must share the same static and dynamic parameters: If a musical parameter is static for one sequence, then it must be static for all sequences sonified; if a musical parameter is dynamic for one sequence, then it must be dynamic for all sequences sonified.

Example: If timbre is a static musical parameter for one sequence being sonified, then this must be true for all sequences. If frequency, dynamic, and spatial location are the sole dynamic parameters for a sequence in sonification (i.e. these are the only parameters that the sequence controls), then they must be the sole dynamic parameters for all sequences sonified.

Rationale: It could lead to confusion and obscure the relationship between the sequences if, for example, frequency was not used as a dynamic parameter for one of the sequences, but was used for all the rest.

5.2.4 Criterion: Shared Values for Static Parameters

Explanation: If a static parameter is attributed a certain value for a particular sequence, then this must be true for all sequences. An exception is made for timbre and spatial location, with criteria to follow.

Example: If a static attack value of 0.008 seconds is used for one sequence in wavetable synthesis, then this same value must be used for all sequences sonified.

Rationale: We are comparing sequences that are composed of integers or sets of integers, and we do not want the integers in one sequence to differ too much in their sound from those in another sequence – we want them to remain recognizable as the same integer, yet we want to be able to distinguish which sequence the integer belongs to.

5.2.5 Criterion: Shared Mapping of Dynamic Parameters

Explanation: If an integer is attributed a certain value of a dynamic parameter for one sequence, then it must be attributed the same value in all sequences sonified. An exception is made for timbre and spatial location, which can be individual for each sequence sonified according to certain criteria that follow.

Example: Let our dynamic parameters in a sonification be frequency, attack and release. If the integer 1 is attributed the frequency 550 Hz in one sequence, and

attack and release values of 0.005 and 0.008 respectively, then this must be true for all sequences sonified.

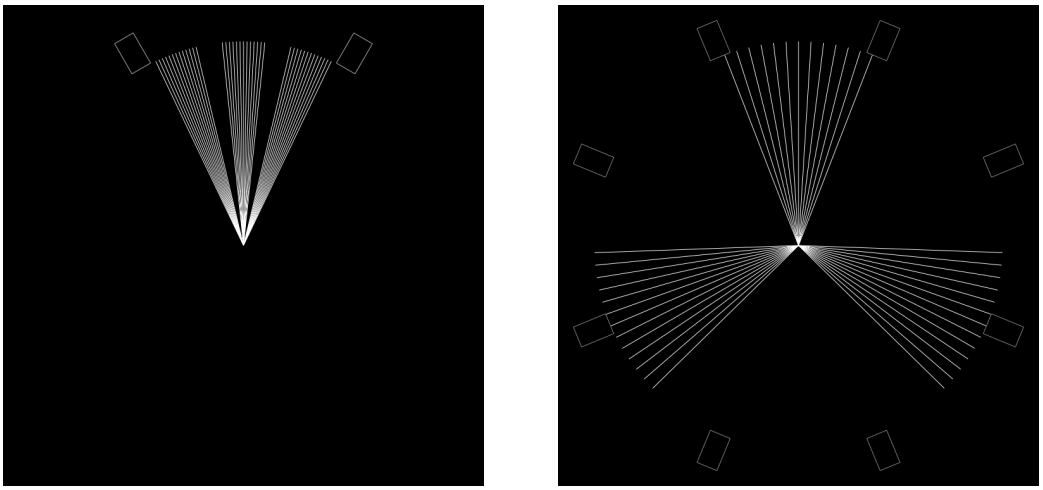
Rationale: We are interested in researching the relationship between the sequences, and the relationship between them could be obfuscated if dynamic parameters are mapped differently for various sequences in the same sonification. Nonetheless, we also do not want the sequences overlapping spatially, as it would then be difficult to tell which sequence an integer belonged to. We would also like to be able to use timbre in a way similar to the way one might use colors in a visual graph – in order to differentiate sequences.

5.2.6 Criterion: Spatialization and a Shared Unit Interval

Explanation: If spatial location is used as a dynamic parameter in a sonification, then the spatial span and direction (clockwise or counterclockwise) of the unit interval must be the same for all sequences sonified. Each sequence is attributed the full range of integers that occurs, from the smallest min of a_n or A_n to the largest max of a_n or A_n that occurs among all sequences sonified, regardless of whether these largest or smallest integers are utilized in a particular sequence. The previous equidistance criterion still applies when spatialization is a dynamic parameter.

Example: If we wanted to simultaneously sonify three GFS $< 0, 1, 1, 2, 3 >$, $< 2, 1, 3, 4, 7 >$ and $< 3, 2, 5, 7, 12 >$ with spatial location as a dynamic parameter, then the first thing we would do is take the full span of integers required (0 through 12) and space them equidistantly in three non-overlapping groups (one group per sequence). The following images show two possibilities, on the left for a stereo setup and on the right for an eight-channel setup. Just as before, the speakers are represented by rectangles, and the line segments represent the path of the sound from the simulated point of origin to the listener (sitting in the center where the line segments converge) for the integers 0 through 12:

Figure 64: A shared unit interval between all sequences.



Regardless of which integers are used in a given sequence, each sequence is attributed the full range of integers required in sonification (here 0 through 12) in order to preserve symmetry and equidistance between the sequences.

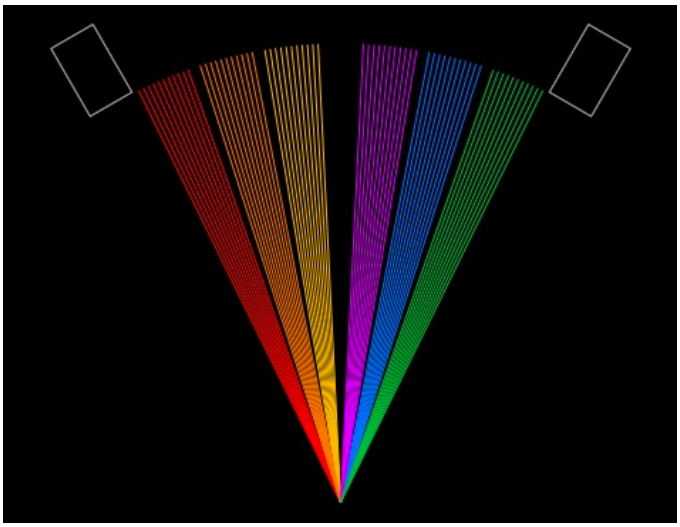
Rationale: Our primary goal is to research the relationship between the sequences, and their relationship could be obfuscated if the unit interval were to vary for different sequences in the same sonification.

5.2.7 Criterion: Equidistance

Explanation: The sequences must be equidistantly placed in spatialization and should not overlap. An exception can be made when the composer desires to highlight some relationship or draw a contrast between groups of sequences by including an additional gap between these groups. In this case, each group must contain the same number of sequences, and each sequence in a group must be equidistantly spaced.

Example: If we had three sequences, then we could place one at $-30°$, one at $0°$, and the other at $30°$. If we had six signature sequences in which R was φ^3, φ^2, φ and Φ, Φ^2, Φ^3, respectively, then the composer might choose to use a gap in between the sequences where $R < 1$ and those where $R > 1$ in order to provide a degree of distinction. Then, the arrangement of the integers in space might appear as follows, with those where $R < 1$ on the left and those where $R > 1$ on the right:

Figure 65: Equidistant placement of sequences with gap between groups.



Rationale: Spatially, it would be best if the sequences remained unobstructed by each other, so that we could clearly differentiate between the sequences. Otherwise, it could be difficult to tell which sequence an integer belonged to. In some cases it is desirable to group sequences together spatially in order to highlight certain elements they have in common in contrast to another group. Each group should have the same number of sequences in order to preserve spatial symmetry.

5.2.8 Criterion: Gap Between Sequences

Explanation: If spatial location is used as a dynamic parameter in a sonification, then the gap between the sequences sonified must be larger than the span of the unit interval.

Example: It can be seen in the two images above that the gap between the sequences is larger than that of the unit interval described in the previous criterion.

Rationale: Each sequence should retain its proper location according to the equidistance criterion and not overlap with the others, which could lead to confusing the integers in one sequence for those in another.

5.2.9 Criterion: Spatial Bilateral Symmetry

Explanation: Bilateral symmetry must be preserved in the spatialization of multiple sequences.

Example: As can be seen in the images shown above, with the listener facing forward, bilateral symmetry is preserved in the points of origin of the integers in space.

Rationale: The clearest and most beautiful presentation is that in which bilateral symmetry is utilized.

5.2.10 Criterion: Timbre

Explanation: Each sequence may be attributed its own timbre (or timbres if it is a dynamic parameter). If two or more sequences share the same timbre or timbres, however, then all the others must as well. As with the equidistance criterion, an exception can be made when grouping sequences together, in which there must be an equal number of sequences in each group. In such a case, each group may be attributed its own timbre (or timbres if it is a dynamic parameter). A sense of good judgment and taste as well as a practical understanding of psychoacoustics will have to be used in the selection of the timbres.

Example: A different grain could be attributed to each sequence when using granular synthesis. Alternatively, if using two groups of signature sequences – one in which $R < 1$ and another group in which $R > 1$ – then one can attribute one grain to one group and a different grain to another group.

Rationale: Just as spatial location, timbre can help in differentiating between the sequences when they occur simultaneously similar to the way color or different characters might be used in a visual graph. Similarly, timbre can be useful in differentiating between groups of sequences with contrasting characteristics.

6 Sonifications of Mathematical Objects Fundamentally Related to FS

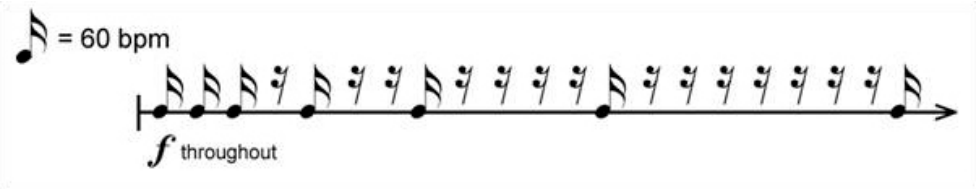
6.1 The Generalized Fibonacci Sequence

As we saw in the works of other composers, reversal is a common artifice used in part to overcome the rapid increase in the values of GFS. The first work presented in this section embraces this rapid increase.

6.1.1 *Fibonacci-Type Sequence* (2005)

In Leipzig, I completed an aleatoric work entitled *Fibonacci-Type Sequence* (for a percussion instrument of indefinite pitch) that utilizes Fibonacci pulse summation in the simplest, most direct manner possible [85]. The following was given as an example of a possible realization in the score:

Figure 66: Theoretical example of *Fibonacci-Type Sequence*.



Tempo and dynamic level can be chosen by the performer (but must remain constant throughout), as can the G_n sonified. The performance may be as long as desired. The first performance, which I executed in Leipzig, lasted nearly 21 hours. Some electronic means of synchronization is required for realization.

6.2 The Golden Ratio

We saw before that GR was rarely utilized directly by composers; we only saw two direct applications (Nørgård and Chowning), one of which was in the electronic medium.

6.2.1 *Golden Ratio Powers* (2012)

In Santa Barbara, I completed an aleatoric electronic work intended as an installation entitled *Golden Ratio Powers* [86], in which consecutive powers of GR are sonified simultaneously in the temporal domain. The instructions for the installation are reproduced below in their entirety:

> These instructions describe an electro-acoustic installation that could be realized outdoors or indoors. Any type of speaker can be used, but all speakers used must be of the same type. Any software synthesis language with good temporal resolution could be used to realize it. The work is

intended to run continuously from the time before the listeners enter the area of the installation until after they have exited.

1. Choose a number of loudspeakers 2 or greater. The speakers are numbered starting at 0: s(0), s(1), s(2), ..., s(n), etc.

2. Arrange the speakers equidistantly in order in a straight line, at least 15 centimeters apart and no more than 12 meters, measured from the center of the speakers. The speakers should remain unobstructed from other objects in between them.

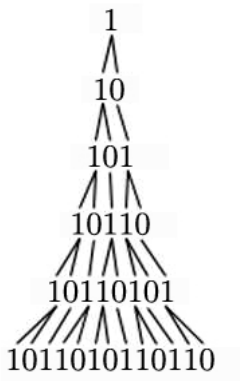
3. A unit duration d is chosen, which must be at least .095 seconds.

4. For each speaker s(n), a single grain occurs continually at the interval of $\Phi^n * d$, where Φ is equal to (1 + sqrt(5))/2. This grain is exactly the same for each speaker – it utilizes the same samples and has the same amplitude. Speakers may start at different times, but all speakers must be producing grains by the time the first listener enters and continue until the last listener has left.

6.3 The Golden String

Another well-known sequence closely related to FS and GR, briefly mentioned in section 3.6.2, is the golden string [87], also known as the rabbit sequence or the infinite Fibonacci word. The sequence can be calculated using the following tree structure, in which the following iterative process is followed in replacing each successive row based on the top: 1 → 10, 0 → 1. This process can be illustrated as follows:

Figure 67: Golden string tree structure.

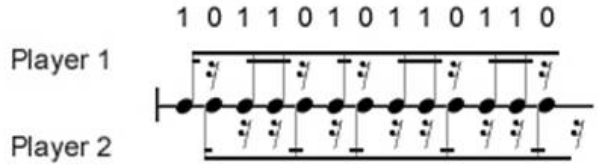


The limit of this sequence is the golden string. It is fractal in that it contains copies of itself, as can be deduced from the iterative process described above. As can be seen, the length of each row is given by F_n, and there are F_{n-1} 1s and F_{n-2} 0s.

6.3.1 *The Rabbit Sequence* (2003)

While living in Munich, I completed a sonification of the golden string for percussionists playing "two percussion instruments of indefinite pitch with crisp attacks and a fast decay" entitled *The Rabbit Sequence* [88]. The following explanation is given in the score for how the work is realized:

Figure 68: *The Rabbit Sequence* in standard notation.

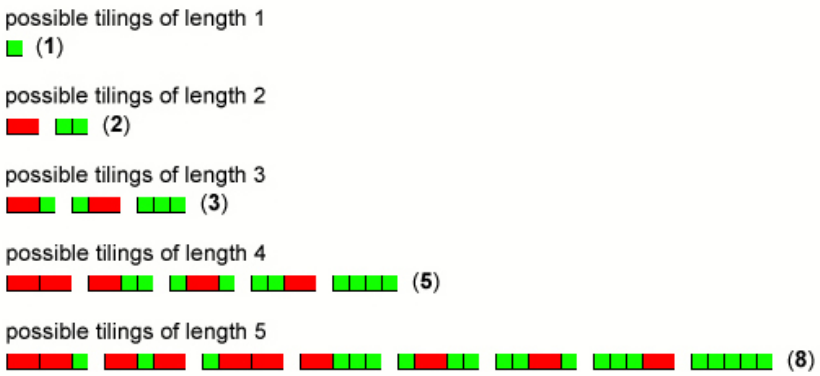


The composition is aleatoric in that the performers are free to make certain decisions. The number of attacks per minute can be freely chosen, but there must be at least 155 attacks per minute. Both parts are to be played at the same dynamic level between *mezzoforte* and *triple forte*. A Fibonacci number of attacks are to be played. Various methods are given for realization, taking advantage of the fractal nature of the sequence to make performance easier for the percussionists. The work is to be played with a click track.

6.4 Fibonacci Tilings

The number of possible one-dimensional tilings of length n using tiles of length 2 and 1, where $n > 0$, is given by FS [89]:

Figure 69: Fibonacci tilings of lengths 1 through 5.



These tilings can be sonified in various manners.

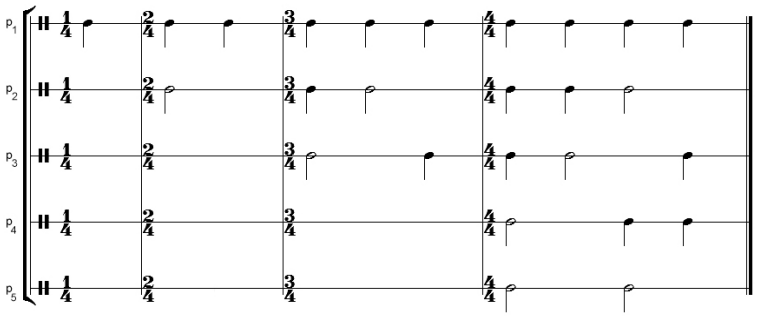
6.4.1 *Fibonacci Tilings no. 1* (2010)

My composition *Fibonacci Tilings no. 1* was written at the 14th International Conference on Fibonacci Numbers and Their Applications in Morelia, Mexico [90]. It was inspired by a lecture on tilings given there by Arthur Benjamin. Two versions of the work exist: one for unpitched instruments and one for pitched instruments. The instructions in the score are as follows:

> 1. Choose a Fibonacci number F_n of instrumentalists – players $p(1)$, $p(2)$, $p(3)$, ..., $p(F_n)$.
>
> 2. Generate all tilings as described above up to length n (all tilings for lengths $1, 2, 3, \ldots, n$).
>
> 3. Rhythmically, each tile of length 2 gets 2 pulses, those of length 1 get 1 pulse.

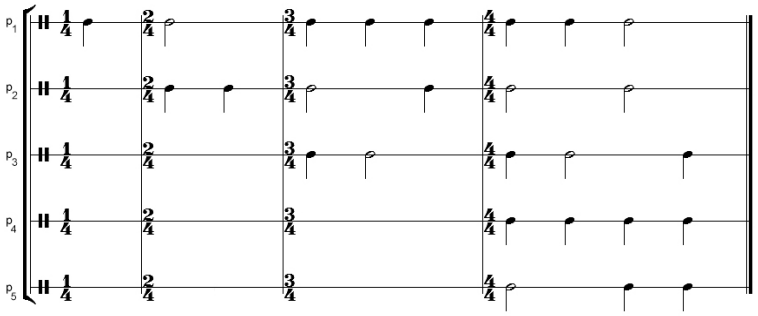
An example is then given of a possible realization:

Figure 70: Theoretical realization of *Fibonacci Tilings no. 1*.



All tilings of a given length are heard simultaneously, and the tilings for lengths $< 1, 2, 3, \ldots, n >$ are heard successively – played in increasing length – as seen in the example above. Players p(1) through p(F_m) play tilings of length m, but it does not matter who plays what tiling. So the following realization would also be acceptable:

Figure 71: Another possible realization of *Fibonacci Tilings no. 1*.



The players are placed equidistantly in a line. Tempo can be freely chosen. All notes are either attributed the same dynamic level, or the tiles of length 2 are attributed a certain dynamic level and those of length 1 a different level. For the unpitched version, performers either use identical instruments, or they all use the same two instruments: one instrument for tiles of length 1, and one for those of length 2. For the pitched version, the tunings described in the criteria are utilized.

6.5 The Wythoff and Stolarsky Arrays

Two arrays closely related to FS and GR are the Wythoff and Stolarsky arrays [91]. In both of these arrays, each row is a G_n, and every integer occurs once and only once in the arrays.

The Wythoff array can be created by taking the non-negative integers $\mathbb{Z}^*$ and the Beatty sequence $\lfloor \Phi(\mathbb{Z}^* + 1) \rfloor$ [92] as starting points – or seeds – for rows of G_n as follows:

Figure 72: The Wythoff array.

$\mathbb{Z}^*$	$\lfloor \Phi(\mathbb{Z}^* + 1) \rfloor$											
0	1	1	2	3	5	8	13	21	34	55	89	→
1	3	4	7	11	18	29	47	76	123	199	322	→
2	4	6	10	16	26	42	68	110	178	288	466	→
3	6	9	15	24	39	63	102	165	267	432	699	→
4	8	12	20	32	52	84	136	220	356	576	932	→
5	9	14	23	37	60	97	157	254	411	665	1076	→
6	11	17	28	45	73	118	191	309	500	809	1309	→
7	12	19	31	50	81	131	212	343	555	898	1453	→
8	14	22	36	58	94	152	246	398	644	1042	1686	→
9	16	25	41	66	107	173	280	453	733	1186	1919	→
10	17	27	44	71	115	186	301	487	788	1275	2063	→
11	19	30	49	79	128	207	335	542	877	1419	2296	→
12	21	33	54	87	141	228	369	597	966	1563	2529	→
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↘

The portion in black is the Wythoff array.

In the Stolarsky array, the first integer in each row k is the lowest that has not yet occurred in any row above. The integer that follows k is given by $[\Phi * k]$:

Figure 73: The Stolarsky array.

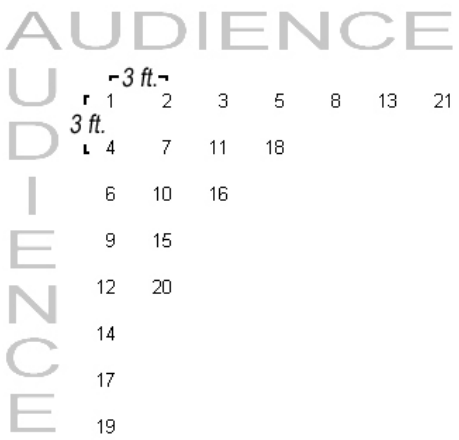
k	$[\Phi * k]$									
1	2	3	5	8	13	21	34	55	89	→
4	6	10	16	26	42	68	110	178	288	→
7	11	18	29	47	76	123	199	322	521	→
9	15	24	39	63	102	165	267	432	699	→
12	19	31	50	81	131	212	343	555	898	→
14	23	37	60	97	157	254	411	665	1076	→
17	28	45	73	118	191	309	500	809	1309	→
20	32	52	84	136	220	356	576	932	1508	→
22	36	58	94	152	246	398	644	1042	1686	→
25	40	65	105	170	275	445	720	1165	1885	→
27	44	71	115	186	301	487	788	1275	2063	→
30	49	79	128	207	335	542	877	1419	2296	→
33	53	86	139	225	364	589	953	1542	2495	→
35	57	92	149	241	390	631	1021	1652	2673	→
38	61	99	160	259	419	678	1097	1775	2872	→
41	66	107	173	280	453	733	1186	1919	3105	→
43	70	113	183	296	479	775	1254	2029	3283	→
46	74	120	194	314	508	822	1330	2152	3482	→
48	78	126	204	330	534	864	1398	2262	3660	→
51	83	134	217	351	568	919	1487	2406	3893	→
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↘

A number of integer sequences can be derived from these arrays. The sequence that gives the row in which an integer n occurs in the Wythoff array is called the horizontal para-Fibonacci sequence; the sequence that gives the column is called the vertical para-Fibonacci sequence (A019586 and A035614 in [2]). One can create two more similar sequences by applying the same principle to the Stolarsky array (A098861 and A098862 in [2]).

6.5.1 *The Wythoff Array* (2005)

In my instrumental composition *The Wythoff Array* [93], written in Leipzig, the Wythoff array is spatialized. The work can be played on either a Fibonacci number of instruments of indefinite pitch, or a Fibonacci number of instruments of definite pitch that can be tuned to a high degree of accuracy – in both cases every player plays the same instrument. A Fibonacci number of players is required, and the performers and audience are arranged as in the following diagram from the score:

Figure 74: Audience and performers in *The Wythoff Array*.



At least three feet must be between the performers, measured from the center of each performer. The audience stands or sits at the edge of the array. The players then simply play through in order 1 through F_n with the same articulation and same dynamic level. Some electronic means of synchronization is required, and electronic renderings of the work are acceptable as well.

Following this composition, a parallel sonification was carried out using the Stolarsky array [94].

6.5.2 *Stolarsky Array no. 2* (2012)

A different approach to the sonification of the Stolarsky array is taken in *Stolarsky Array no. 2* [95]. In this composition, the two above-mentioned integer sequences that give the rows and the columns in which n occurs are sonified simultaneously. The composition is in stereo, with one sequence on each speaker. Granular synthesis with a resonance filter is used, and each integer is assigned its own resonance center frequency and resonance bandwidth. The temperament used is $\varphi^8 + 1$. The sound source for the grain was some unused lump charcoal that I had recorded colliding. The loudness level was made to be static at *fortissimo* so as not to obscure one sequence or the other, seeing that their ranges differ significantly (the range of integers is 0 to 54 for A098861 and 0 to 10 for A098862). The abridged score follows:

Figure 75: *Stolarsky Array no. 2.*

Collection XIV

Stolarsky Array no. 2
Casey Mongoven
April 30, 2012

classification of work: audio-visual
synthesis engine: Csound show Csound orchestra
synthesis technique: granular synthesis with resonance filter

description of sequences sonified

A098861 gives number of row in Stolarsky array that contains n 144 members used

A098862 gives number of column in Stolarsky array that contains n 144 members used

global static parameters:

sequence offset: 1
temperament: phi^{8} + 1
frequency orientation: descending
number of channels: 2
duration of single member of sequence: 0.14 seconds
piece duration: 20.16 seconds
grain: raw charcoal 3
loudness: ff

individual static parameters:

location:
−30°
30°

global dynamic parameters:

A098861

integer	resonance center frequency Hz	resonance bandwidth Q factor
0	1930.000000	7.692308
1	1889.773828	7.758621
2	1850.386073	7.826087
3	1811.819260	7.884737
4	1774.056279	7.964602
5	1737.080376	8.035714
6	1700.875145	8.108108
7	1665.424525	8.181818
⋮	⋮	⋮
52	645.488338	13.846154
53	632.034698	14.062500
54	618.861467	14.285714

Figure 75: *Stolarsky Array no. 2* (continued).

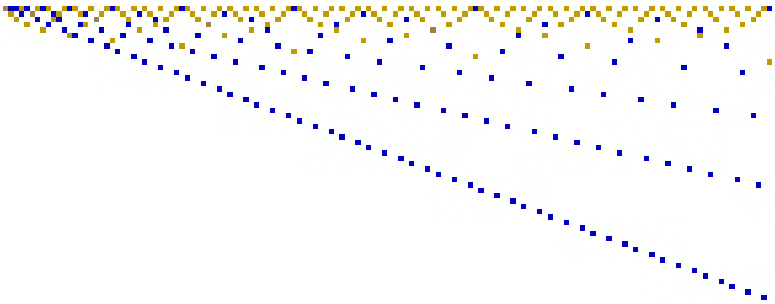
A098862

integer	resonance center frequency Hz	resonance bandwidth Q factor
0	1930.000000	7.692308
1	1889.773828	7.758621
2	1850.386073	7.826087
3	1811.819260	7.884737
4	1774.056279	7.964602
5	1737.080376	8.035714
6	1700.875145	8.108108
7	1665.424525	8.181818
8	1630.712788	8.256881
9	1596.724532	8.333333
10	1563.444679	8.411215

144 values used of Sloane's A098861: 0, 0, 0, 1, 0, 1, 2, 0, 3, 1, 2, 4, 0, 5, 3, 1, 6, 2, 4, 7, 0, …,52, 32, 12, 53, 20, 33, 54, 0

144 values used of Sloane's A098862: 0, 1, 2, 0, 3, 1, 0, 4, 0, 2, 1, 0, 5, 0, 1, 3, 0, 2, 1, 0, 6, …, 0, 1, 3, 0, 2, 1, 0, 10

graph of sequences: 1) A098861, 144 values; 2) A098862, 144 values.



A similar sonification of the Wythoff array is *Wythoff Array no. 2* [96]. A sonification of the Stolarsky and Wythoff arrays together is *Stolarsky and Wythoff Arrays no. 2* [97].

6.6 The Monotonic Justified Array

Another array of interest is the monotonic justified array, which is an ordering of all positive G_n – i.e. of all possible G_n containing positive integers. The array is called justified because, in every G_n, $G_1 > G_2 \geqslant 0$. The first two columns in such an array can be represented as an ordered pair (a_i, b_i), where i represents the row index. When representing a single row, we can drop the subscript i and simply write (a, b), where $a > b \geqslant 0$. The ordering of G_n is determined by the following rule [98]:

$$(a, b) < (c, d) \text{ if and only if there exists } N \text{ such that}$$
$$aF_n + bF_{n+1} < cF_n + dF_{n+1} \text{ for } n \geqslant N.$$

The north-west corner of the array is as follows:

Figure 76: The monotonic justified array.

1	0	1	1	2	3	5	8	13	21	34	55	→
2	0	2	2	4	6	10	16	26	42	68	110	→
3	0	3	3	6	9	15	24	39	63	102	165	→
2	1	3	4	7	11	18	29	47	76	123	199	→
4	0	4	4	8	12	20	32	52	84	136	220	→
3	1	4	5	9	14	23	37	60	97	157	254	→
5	0	5	5	10	15	25	40	65	105	170	275	→
4	1	5	6	11	17	28	45	73	118	191	309	→
6	0	6	6	12	18	30	48	78	126	204	330	→
3	2	5	7	12	19	31	50	81	131	212	343	→
5	1	6	7	13	20	33	53	86	139	225	364	→
7	0	7	7	14	21	35	56	91	147	238	385	→
4	2	6	8	14	22	36	58	94	152	246	398	→
6	1	7	8	15	23	38	61	99	160	259	419	→
8	0	8	8	16	24	40	64	104	168	272	440	→
5	2	7	9	16	25	41	66	107	173	280	453	→
7	1	8	9	17	26	43	69	112	181	293	474	→
4	3	7	10	17	27	44	71	115	186	301	487	→
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↘

We can derive a sequence from this that gives the smallest row that contains an integer n (A199087 in [2]). Excluding the columns (a_i, b_i), we can derive a sequence that gives the lowest column index that contains n in the smallest row that contains n (A199088 in [2]).

6.6.1 *Monotonic Justified Array Vertical no. 7* (2012)

The composition *Monotonic Justified Array Vertical no. 7* is a sonification of the sequence that gives the smallest row in which n occurs in the monotonic justified array. In this composition, the dynamic parameters are loudness, simulated location and timbre. Pure granular synthesis (with no resonance filter) was used. Two grains of 445 samples each (at 96 kHz) are utilized – derived from the sound of burning charcoal – and additive synthesis is used to "blend" between the grains so that the timbre gradually changes from the sound of one grain to another from the lowest integer 1 to the highest integer 400. The sonification has a clockwise orientation, meaning that lower integers come from the left side and higher from the right. The point of termination in this particular composition was chosen to be based on sequence A059769 in [2].

Figure 77: *Monotonic Justified Array Vertical no. 7.*

Collection XIV

Monotonic Justified Array Vertical no. 7
Casey Mongoven
May 1, 2012

classification of work: audio-visual
synthesis engine: Csound show Csound orchestra
synthesis technique: granular synthesis

description of sequence sonified

A199087 gives smallest row that contains an integer n in the monotonic justified array of generalized Fibonacci sequences **659 members used**

global static parameters:

sequence offset: 1
number of channels: 2
duration of single member of sequence: 0.037 seconds
piece duration: 24.383 seconds

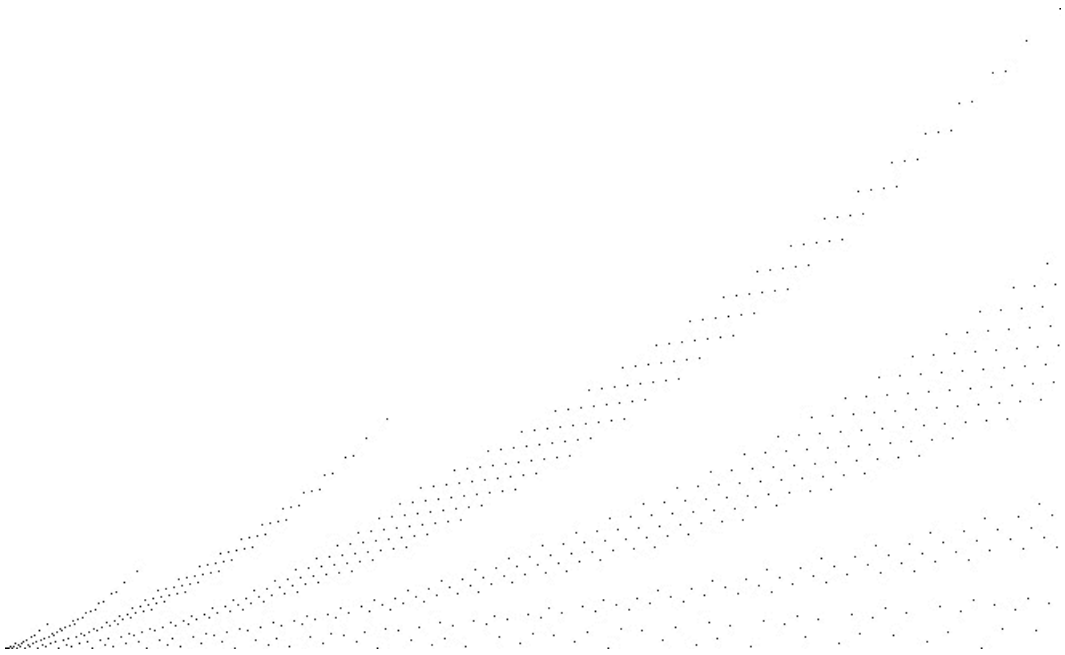
global dynamic parameters:

integer	loudness	simulated loc. deg.	grain
1	pp	-26.016	burn._charc._8*.9667+burn._charc._9*.0333
2	pp	-25.886	burn._charc._8*.9641+burn._charc._9*.0359
3	pp	-25.755	burn._charc._8*.9616+burn._charc._9*.0384
4	pp	-25.625	burn._charc._8*.9591+burn._charc._9*.0409
5	pp	-25.495	burn._charc._8*.9565+burn._charc._9*.0435
6	pp	-25.364	burn._charc._8*.9540+burn._charc._9*.0460
7	pp	-25.234	burn._charc._8*.9514+burn._charc._9*.0486
8	pp	-25.103	burn._charc._8*.9489+burn._charc._9*.0511
9	pp	-24.973	burn._charc._8*.9464+burn._charc._9*.0536
10	pp	-24.843	burn._charc._8*.9438+burn._charc._9*.0562
⋮	⋮	⋮	⋮
361	f	20.930	burn._charc._8*.1579+burn._charc._9*.8421
380	f	23.408	burn._charc._8*.1134+burn._charc._9*.8866
400	f	26.016	burn._charc._8*.0624+burn._charc._9*.9376

659 values used of Sloane's A199087: 1, 1, 1, 2, 1, 2, 4, 1, 3, 2, 4, 5, 1, 6, 3, 2, 8, 4, 9, 5, 1, …, 29, 202, 84, 167, 228, 64, 190, 400

Figure 77: *Monotonic Justified Array Vertical no. 7* (continued).

graph of sequence A199087, 659 values



6.7 Zeckendorf Representations and Generalizations

As mentioned earlier, Zeckendorf's theorem guarantees $n \in \mathbb{N}$, where $n > 0$, can be represented as

$$Z(n) = \sum_{j=2}^{M} c_j F_j,$$

where $c_j \in \{0, 1\}$ and $c_j c_{j+1} = 0$ [3]. This Zeckendorf representation is called minimal in that the least number of Fibonacci numbers (or bits) are used. We define three functions from this:

$$Z_b(n) = \ < c_j \mid F_j \in Z(n) >$$

$$Z_i(n) = \ \{j \mid F_j \in Z(n)\}$$

$$Z_s(n) = \ \{F_j \mid F_j \in Z(n)\}$$

Thus, for example,

$$Z(17) = 1F_7 + 0F_6 + 0F_5 + 1F_4 + 0F_3 + 1F_2.$$

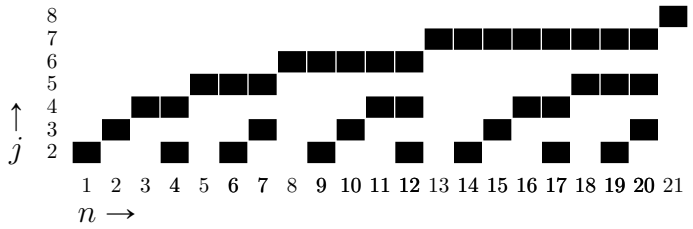
$Z_b(17) = < 1, 0, 0, 1, 0, 1 >$, usually written as the bit string 100101 – the sequence of coefficients c and conventional Zeckendorf representation of 17.

$Z_i(17) = \{2, 4, 7\}$, the set of Fibonacci indices.

$Z_s(17) = \{1, 3, 13\}$, the set of Fibonacci numbers summing to 17.

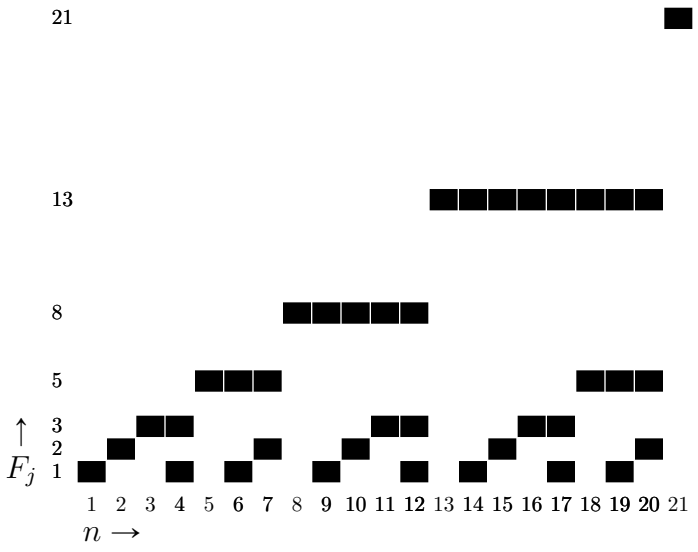
$Z_i(n)$, or $Z_b(n)$ (where a black point indicates a 1), can be graphed as follows:

Figure 78: Graph of $Z_i(n)$ or $Z_b(n)$.



$Z_s(n)$ can be graphed as:

Figure 79: Graph of $Z_s(n)$.



These two functions are suggestive of different compositional approaches to mapping the Zeckendorf representations to musical parameters: one can either choose to sonify the representations using the indices of the Fibonacci numbers $Z_i(n)$ or, alternatively, the Fibonacci numbers $Z_s(n)$.

Integer sequences can be derived from counting the number of 1s or 0s in $Z_b(n)$ (A007895 and A102364 in [2]), or the number of runs of equal bits (A104324 in [2]).

$Z_b(17) = < 1, 0, 0, 1, 0, 1 >$, for example, has 5 runs of equal bits: $< 1 >$, $< 0, 0 >$, $< 1 >$, $< 0 >$ and $< 1 >$.

As mentioned, no consecutive 1s occur in $Z_b(n)$. Zeckendorf representations are not the only possible representations of integers as sums of unique Fibonacci numbers. Brown introduced the dual Zeckendorf representation in [99], a representation in which no consecutive 0s are allowed, showing that $n \in \mathbb{N}$ can also be represented as

$$\sum_{j=2}^{M} c_j F_j,$$

where $c_j \in \{0, 1\}$ and $c_j + c_{j+1} \geqslant 1$. The first 13 Zeckendorf and dual Zeckendorf representations are shown below in their binary form:

Figure 80: Zeckendorf and dual Zeckendorf representations.

n	**Zeckendorf representation**	**dual Zeckendorf representation**
1	1	1
2	10	10
3	100	11
4	101	101
5	1000	110
6	1001	111
7	1010	1010
8	10000	1011
9	10001	1101
10	10010	1110
11	10100	1111
12	10101	10101
13	10000	10110

The Zeckendorf and dual Zeckendorf representations are also sometimes referred to as minimal and maximal Fibonacci representations, respectively. Similar count-based integer sequences can be derived from the dual Zeckendorf representation as well: counting the number of 0s (A117479 in [2]), 1s (A112310) and runs (A104325).

Just as with FS, it is possible to represent all integers as sums of non-adjacent Lucas numbers [100]:

$$A(n) = \sum_{j=0}^{M} c_j L_j,$$

where $c_j \in \{0, 1\}$ and $c_j c_{j+1} = 0$. These are called minimal Lucas representations, and here it is also possible to create a maximum representation in which no consecutive 0s are allowed:

Figure 81: Minimal and maximal Lucas representations.

n	**minimal Lucas representation**	**maximal Lucas representation**
1	10	10
2	1	1
3	100	11
4	1000	110
5	1010	101
6	1001	111
7	10000	1011
8	10010	1110
9	10001	1101
10	10100	1111
11	100000	10110

6.7.1 *Zeckendorf Representations no. 17* (2005)

The work *Zeckendorf Representations no. 17* exists in both eight-channel and stereo form [101]. Like the 16 works preceding it, $Z_i(n)$ was sonified as opposed to $Z_s(n)$. $Z_i(n)$ was mapped to two parameters: frequency and spatial location. A total of $F_{14} = 377$ representations are sonified. Higher integers are represented by higher pitches; the frequency orientation of the work is thus ascending. Each Fibonacci number is attributed the dynamic level *forte*. The spatialization is clockwise, moving from lower integers on the left to higher ones on the right. Delay between the right and left speakers was used in the stereo version in addition to panning to simulate the location of the integers. VBAP (vector-based amplitude panning) was used in the eight-channel version. In this case, no additional attack was added to the envelope of the dynamically compressed, low-pass filtered, harpsichord sound; a release of 0.008 seconds was nonetheless added to the end of each note. Here is an abridged score of the work:

Figure 82: *Zeckendorf Representations no. 17*.

Collection VIII

Zeckendorf Representations no. 17
Casey Mongoven
November 9, 2008

description of sequence: `Zeckendorf representation of n`
offset: `1`
number of members used: `377`
number of channels: `2 (or 8)`
piece length: `519.9961 seconds`
note value: `1.3793 seconds`

Figure 82: *Zeckendorf Representations no. 17* (continued).

orientation: ascending
temperament: $\text{phi}^7 + 1$
lowest frequency: 175.45 Hz **highest frequency:** 263.4 Hz
number of unique frequencies: 13
synthesis technique: raw samples with analog dynamic compression, digital reverb and low-pass filter
waves: Sperrhake harpsichord, keys Eb3 to Eb4 (retuned to $\text{phi}^7 + 1$)
dynamic: each voice f
simulated spatial location: left to right through panning and delay (or clockwise with VBAP)
delay: -.00069 to .00069 (8-channel version without delay)
attack: natural
release: .008
synthesis language used: Csound
note: For the 8-channel version, the speakers should be arranged in a circle.

Fibonacci index	location	frequency
2	1	175.45
3	2	181.49
4	3	187.74
5	4	194.21
6	5	200.89
7	6	207.81
8	7	214.97
9	8	222.37
10	9	230.03
11	10	237.96
12	11	246.15
13	12	254.63
14	13	263.40

sequence values used: 1, 10, 100, 101, 1000, 1001, 1010, 10000, …, 101010101001, 101010101010, 1000000000000

graph:



As indicated, this work used an acoustic instrument as an initial sound source: the composer's own Sperrhake harpsichord. Upon deciding to use the harpsichord, some technical considerations of the instrument came into play in composition: 1) it was desirable that each Fibonacci number be represented by a different frequency

and a different string and 2) that the strings used be adjacent to one another. The tuning $\varphi^6 + 1 \approx 1.05573$ is the closest tuning satisfying $\varphi^t + 1$ to the standard tuning of the harpsichord, which is – at least in theory – based on $\sqrt[12]{2} \approx 1.05946$. Nonetheless, in order to avoid too much similarity to the standard tuning with this familiar instrument, the harpsichord was tuned to $\varphi^7 + 1 \approx 1.03444$ instead. This tuning imposed restrictions on how many adjacent strings could be retuned without breaking the strings that needed to be tuned to a higher frequency; in the end, it was determined that using 13 different tones representing 13 different integers would be feasible.

After calculating the number of occurrences of each Fibonacci number in the Zeckendorf representations from $Z(1)$ to $Z(377)$, each tone was recorded individually. The Fibonacci number 1, for example, was played and recorded 144 times – in reality a few times more, as the loudest and quietest notes were selectively removed from the raw recording. Following this, the order of the notes recorded was randomized using a true random number generator [102]; this randomization was carried out in order to "smooth out" natural tendencies the composer may have had in varying the amplitude of each attack in the course of playing, and in order to randomize any ever-so-slight variations in tuning that occurred over the course of the recording process.

It was the composer's goal in the sound design process to render the harpsichord practically unrecognizable, yet retain some of its natural qualities; to this end an analog dynamic compressor, digital reverb, and a digital low-pass Butterworth filter – with the filter cut-off sweeping exponentially from the 3rd harmonic partial of the fundamental frequency of the string used to the 8th over the course of each note – were applied. The effect of the low-pass filter in this work is most significant in transforming the sound of the harpsichord, as it acts to dampen the high frequencies present in the transient – the initial attack – of the harpsichord, a feature that is so characteristic of its sound.

Each note of every chord had to be adjusted by ear individually in order to compensate for some auditory masking and loudness variations that naturally occurred when combining the tones; the goal in this piece was to make each number represented sound at the dynamic level *forte*. To this end, both versions of the work were listened to dozens of times while the composer simultaneously adjusted the amplitude of each individual note.

6.8 Phinary Representations

In addition to using FS as the base of a number system, it is also possible to represent integers as sums of powers of GR [103], i.e. $n \in \mathbb{N}$, where $n > 0$, can be represented as

$$P(n) = \sum_{j=-\infty}^{\infty} c_j \Phi^j,$$

where $c_j \in \{0, 1\}$ and $c_j c_{j+1} = 0$. For example,

$$P(10) = 1\Phi^4 + 0\Phi^3 + 1\Phi^2 + 0\Phi^1 + 0\Phi^0 + 0\Phi^{-1} + 1\Phi^{-2} + 0\Phi^{-3} + 1\Phi^{-4}.$$

We use a radix point to separate those powers greater to or equal to one less than one, so one could write the above as 10100.0101. Because $\Phi^n = \Phi^{n+1}+\Phi^{n+2}$, we can replace any occurrence of $\ldots 100 \ldots$ with $\ldots 011 \ldots$ This means that there are infinitely many representations of n as a sum of powers of GR. We therefore define the minimum phinary representation to be that binary representation containing the least number of 1s. The maximum phinary will be defined as the shortest possible representation containing no consecutive 0s. Phinary representations are closely related to and can be derived from the minimal Lucas representations previously discussed. The first 18 minimum and maximum phinary representations are shown below:

Figure 83: Minimal and maximal phinary representations.

n	**min. phinary representation**	**max. phinary representation**
1	1	1
2	10.01	1.11
3	100.01	11.01
4	101.01	11.1111
5	1000.1001	101.1111
6	1010.0001	111.0111
7	10000.0001	1010.1101
8	10001.0001	1011.1101
9	10010.0101	1101.1101
10	10100.0101	1111.0101
11	10101.0101	1111.111111
12	100000.101001	10101.111111
13	100010.001001	10111.011111
14	100100.001001	11010.110111
15	100101.001001	11011.110111
16	101000.100001	11101.110111
17	101010.000001	11111.010111
18	1000000.000001	101010.101101

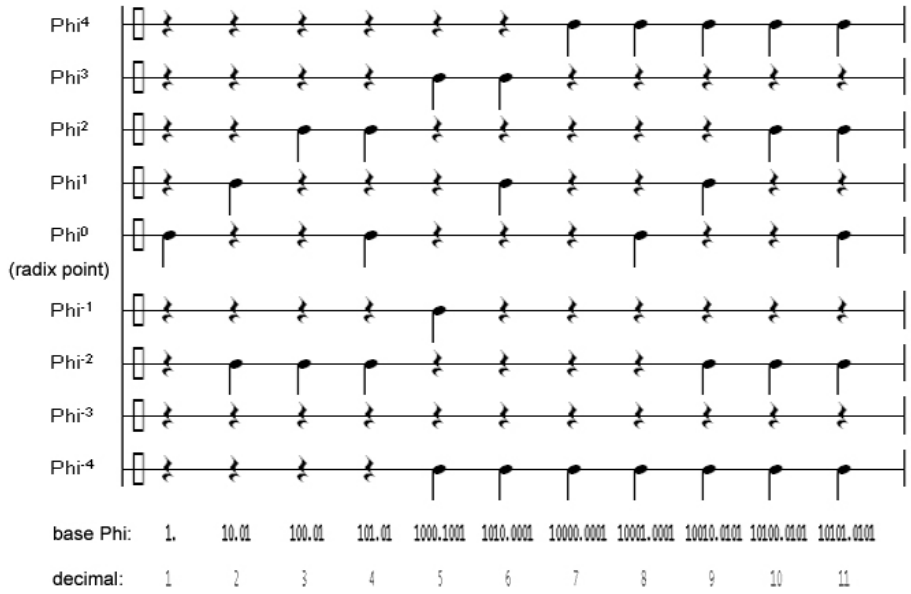
6.8.1 *Base Phi Representations: Base Phi Minimum/Maximum Binary* (2006)

In this pair of works [104, 105], which share the same score, each player is assigned a power of Phi. The work is performed with either the minimal or maximal representations – mixing the two in the same performance is not allowed. Each performer plays a note – the same note – every time his or her assigned power of GR occurs in the integer being expressed. There are three versions of the works: one for percussion instruments of indefinite pitch, one for percussion instruments of definite pitch, and one for pitched instruments (in which some percussion instruments are allowed as well). The tunings allowed are those explained in the criteria section 5.1.12 that are

based on GR.

Each player plays the same type of instrument in every case. Each member of the sequence is given equal duration. The duration of each member is up to the performers to decide, but should be at maximum 0.857 seconds. A point of termination corresponding to LS has to be chosen. The following graphic of a theoretical performance with the minimum representations is provided with the score:

Figure 84: Theoretical performance of *Base Phi Representations: Base Phi Minimum.*



Several options are possible for the spacing of the performers on stage, who always have to be arranged in a straight line: equidistance, equidistance with an extra space representing the radix point, and exponential spacing determined by powers of GR. These works were written in Hamburg and are dedicated to Ron Knott, who has been an enormous help to me in many respects.

6.8.2 *Minimum Phinary no. 6* (2012)

Minimum Phinary no. 6 uses a form of wavetable synthesis where the waveform is first resynthesized using FFT and the partials above the Nyquist frequency are eliminated in order to avoid aliasing [106]. Just as in some of the works utilizing granular synthesis discussed earlier, the source of the waveform is derived from charcoal – this time the crackling of burning charcoal. The point of termination is a Lucas number $L_{14} = 521$.

Figure 85: *Minimum Phinary no. 6.*

Collection XIV

Minimum Phinary no. 6
Casey Mongoven
June 14, 2012

classification of work: audio-visual
synthesis engine: Csound show Csound orchestra
synthesis technique: wavetable with FFT resynthesis

description of sequence sonified

MINPHI minimum representation of n in base of powers of the golden ratio
521 members used

global static parameters:

sequence offset: 1
temperament: phi5 + 1
frequency orientation: ascending
number of channels: 2
duration of single member of sequence: 0.072 seconds
piece duration: 37.512 seconds
grain: burning charcoal 28

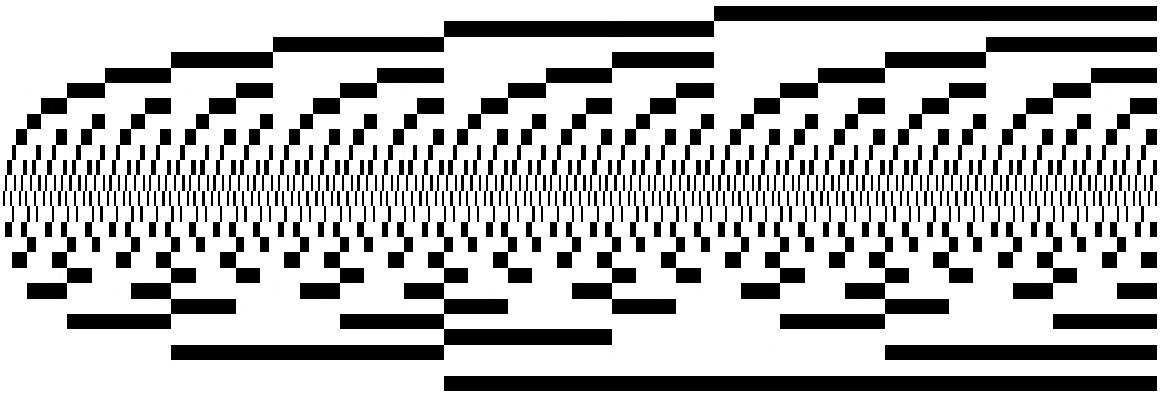
global dynamic parameters:

power of Phi	frequency Hz	loudness	attack s	release s	simulated location degrees
-12	131.000000	mp	0.012300	0.014400	25.808
-11	142.812263	mp	0.012104	0.014171	23.658
-10	155.689636	mp	0.011908	0.013942	21.507
-9	169.728162	mp	0.011713	0.013713	19.356
-8	185.032541	mp	0.011517	0.013483	17.206
⋮	⋮	⋮	⋮	⋮	⋮
10	875.273519	mf	0.007992	0.009358	-21.507
11	954.196883	mf	0.007796	0.009129	-23.658
12	1040.236762	mf	0.007600	0.008900	-25.808

521 values used of sequence MINPHI: 1., 10.01, 100.01, 101.01, 1000.1001, 1010.0001, 10000.0001, 10001.0001, …, 1010101010010.010101010101, 1010101010100.010101010101, 1010101010101.010101010101

Figure 85: *Minimum Phinary no. 6* (continued).

graph of sequence MINPHI, 521 values:



6.9 Sequences Counting Possible Representations

We saw in section 6.7 that the Zeckendorf representation is not always the only possible representation of an integer as a sum of integers from FS. A new category of sequences can be created by counting the number of possible representations of an integer n as sums of distinct integers of various G_n [107]. For instance, if $G_n = < 1, 2, 3, 5, 8, 13, \cdots >$, we can list the number of possible representations of an integer as follows:

Figure 86: Possible representations of n with $G_n = < 1, 2, 3, 5, 8, 13, \cdots >$.

n	# possible reps	reps
1	1	1
2	1	2
3	2	3 = 2+1
4	1	3+1
5	2	5 = 3+2
6	2	5+1 = 3+2+1
7	1	5+2
8	3	8 = 5+3 = 5+2+1
9	2	8+1 = 5+3+1
10	2	8+2 = 5+3+2
11	3	8+3 = 8+2+1 = 5+3+2+1
12	1	8+3+1
13	3	13 = 8+5 = 8+3+2
↓	↓	↓

With some G_n it is not possible to represent certain integers. For instance, if $G_n = < 1, 3, 4, 7, 11, 18, \cdots >$:

Figure 87: Possible representations of n with $G_n = \ < 1, 3, 4, 7, 11, 18, \cdots >$.

n	# possible reps	reps
1	1	1
2	0	–
3	1	3
4	2	3+1 = 4
5	1	4+1
6	0	–
7	2	7 = 4+3
8	2	8 = 4+3+1
9	0	–
10	1	7+3
11	3	11 = 7+4 = 7+3+1
↓	↓	↓

6.9.1 *G(1,3) and G(1,4) Rep Sequences no. 1* (2012)

The first sequence used in this work [108] is based on the number of possible representations of an integer n as a sum of distinct elements of the Lucas sequence beginning $< 1, 3, 4, 7, 11, \cdots >$ (A003263 in [2]) [109]. For example, the integer 8 can be represented in two ways: $7 + 1$ or $4 + 3 + 1$, so $a_8 = 2$. Integers that are not representable (e.g. 2 and 6) are sonified as silence. The second sequence sonified follows the exact same principle, but uses elements of the GFS $< 1, 4, 5, 9, 14, \cdots >$ instead (A103344 in [2]). 1363 members of the first sequence were used, and 1740 of the second. This means that in this case, the first sequence has an earlier point of termination than the second. This was done in order to highlight a certain relationship between these sequences: if one removes all of the 0s from both sequences, which are represented as silence in this sonification, then the sequences are identical. The musical form resulting from the combination of these sequences is therefore a unique type of mensural canon by augmentation.

The synthesis technique used in this composition was granular synthesis with a resonance filter. In the sonification of integer sequences, a grain can be used to represent an integer. This technique can be highly useful because, compared to other synthesis techniques, a much larger number of integers can be heard in a short timespan. If a resonance filter is used to filter a grain, as it is here, then each integer can be attributed its own resonance center frequency. Similarly, each grain can be attributed its own location according to the sequence or its own level of loudness. The grain used was from a viola built by Anne Cole, named "1980" after the year it was made. The note duration here is 0.025 seconds. The unit interval of the tuning is $\varphi^7 + 1$ with ascending frequency orientation. The natural filtered attack of the grain was left unaltered, but a static release value of 0.0089 was used at the end of each note. Location is a static parameter with each sequence placed on an individual loudspeaker. Three global dynamic parameters are utilized: resonance filter center frequency, resonance filter bandwidth and loudness.

Figure 88: *G(1,3) and G(1,4) Rep Sequences no. 1.*

Collection XIII

G(1,3) and G(1,4) Rep Sequences no. 1
Casey Mongoven
March 18, 2012

classification of work: audio-visual
synthesis engine: Csound show Csound orchestra
synthesis technique: granular synthesis with resonance filter

description of sequences sonified

A003263 number of possible representations of n as a sum using distinct elements of the Lucas sequence beginning 1,3,4,7,11,… 1363 members used

A103344 number of possible representations of n as a sum using distinct elements of the Fibonacci-type sequence beginning 1,4,5,9,14,… 1740 members used

global static parameters:

offset: 1
temperament: phi^7 + 1
pitch orientation: ascending
number of channels: 2
note duration: 0.025 seconds
piece duration: 43.5 seconds
grain: cole 1980 24
release: 0.0089 seconds

individual static parameters:

location:
−30°
30°

global dynamic parameters:

A003263

integer	resonance center frequency Hz	resonance bandwidth Q factor	loudness
1	850.000000	8.333333	pp
2	879.275576	8.474576	pp
3	909.559456	8.620690	pp
4	940.886370	8.771930	p
5	973.292241	8.928571	p
⋮	⋮	⋮	⋮
24	1852.076217	13.513514	ff
25	1915.865155	13.888889	ff
26	1981.851103	14.285714	ff

Figure 88: *G(1,3) and G(1,4) Rep Sequences no. 1* (continued).

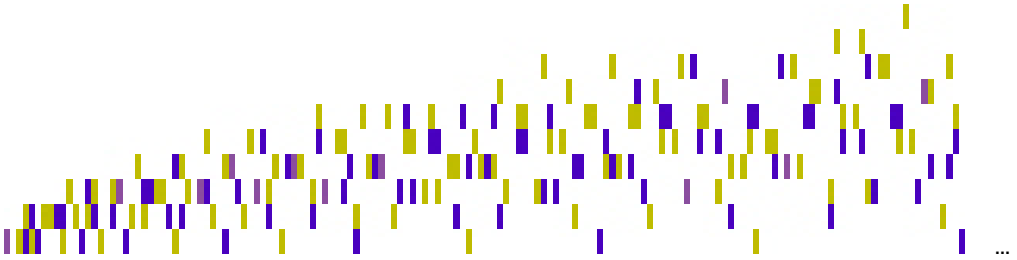
A103344

integer	resonance center frequency Hz	resonance bandwidth Q factor	loudness
1	850.000000	8.333333	pp
2	879.275576	8.474576	pp
3	909.559456	8.620690	pp
4	940.886370	8.771930	p
5	973.292241	8.928571	p
⋮	⋮	⋮	⋮
24	1852.076217	13.513514	ff
25	1915.865155	13.888889	ff
26	1981.851103	14.285714	ff

1363 values used of Sloane's A003263: 1, 0, 1, 2, 1, 0, 2, 2, 0, 1, 3, 2, 0, 2, 3, 1, 0, 3, 3, 0, 2, …, 6, 12, 6, 0, 7, 7, 0, 1

1740 values used of Sloane's A103344: 1, 0, 0, 1, 2, 1, 0, 0, 2, 2, 0, 0, 1, 3, 2, 0, 0, 2, 3, 1, 0, …, 6, 0, 0, 7, 7, 0, 0, 1

graph of sequences: 1) A003263, 1363 values; 2) A103344, 1740 values.



The canonic principle inherent in this work was taken to an even higher level in the eight-channel work *Rep Sequences no. 1* [110], in which eight such sequences are utilized at once, one per speaker.

6.10 Modulus-Based Sequences

We saw in 4.1.1 that FS under a modulus is periodic [111]. We can derive a host of new sequences based on FS modulo m. A first pair of sequences can be obtained from the period of such sequences (all GFS modulo m are periodic due to the limited number of residues modulo m). A second sequence can be obtained from the so-called restricted period, which gives the second occurrence of 0 in such sequences:

Figure 89: Periods and restricted periods of FS modulo m.

m	**FS modulo** m	**period**	**restricted period**
1	$< 0 >$	1	1
2	$< 0, 1, 1 >$	3	3
3	$< 0, 1, 1, 2, 0, 2, 2, 1 >$	8	4
4	$< 0, 1, 1, 2, 3, 1 >$	6	6
5	$< 0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1 >$	20	5
6	$< 0, 1, 1, 2, 3, 5, 2, 1, 3, 4, 1, 5, 0, 5, 5, 4, 3, 1, 4, 5, 3, 2, 5, 1 >$	24	12
7	$< 0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1 >$	16	8
8	$< 0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1 >$	12	6
9	$< 0, 1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 0, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1 >$	24	12
10	$< 0, 1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, \ldots, 9, 3, 2, 5, 7, 2, 9, 1 >$	60	15
11	$< 0, 1, 1, 2, 3, 5, 8, 2, 10, 1 >$	10	10
12	$< 0, 1, 1, 2, 3, 5, 8, 1, 9, 10, 7, 5, 0, 5, \ldots, 1, 4, 5, 9, 2, 11, 1 >$	24	12
13	$< 0, 1, 1, 2, 3, 5, 8, 0, 8, 8, 3, 11, 1, \ldots, 0, 5, 5, 10, 2, 12, 1 >$	28	7

We can then count the number of unique residues as well as the number of absent residues in such sequences:

Figure 90: Unique and absent residues in FS modulo m.

m	F_n modulo m	**unique**	**absent**
1	$< 0 >$	1	0
2	$< 0, 1, 1 >$	2	0
3	$< 0, 1, 1, 2, 0, 2, 2, 1 >$	3	0
4	$< 0, 1, 1, 2, 3, 1 >$	4	0
5	$< 0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1 >$	5	0
6	$< 0, 1, 1, 2, 3, 5, 2, 1, 3, 4, 1, 5, 0, 5, 5, 4, 3, 1, 4, 5, 3, 2, 5, 1 >$	6	0
7	$< 0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1 >$	7	0
8	$< 0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1 >$	6	2
9	$< 0, 1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 0, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1 >$	9	0
10	$< 0, 1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, \ldots, 9, 5, 4, 9, 3, 2, 5, 7, 2, 9, 1 >$	10	0
11	$< 0, 1, 1, 2, 3, 5, 8, 2, 10, 1 >$	7	4
12	$< 0, 1, 1, 2, 3, 5, 8, 1, 9, 10, 7, 5, 0, 5, 5, 10, 3, 1, 4, 5, 9, 2, 11, 1 >$	11	1
13	$< 0, 1, 1, 2, 3, 5, 8, 0, 8, 8, 3, 11, \ldots, 8, 5, 0, 5, 5, 10, 2, 12, 1 >$	9	4
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$

6.10.1 *Absent and Unique Residues no. 3* (2012)

In this work [112], the sequence of unique residues is sonified simultaneously with the sequence of absent residues.

The sequence of unique residues is placed on the left speaker and the sequence of absent residues on the right. As seen in an earlier work, a single grain created from the sound of pieces of raw lump charcoal colliding together was used as a sound source – a light percussive sound rich mostly in higher frequency components. The only dynamic parameter in this work is loudness, ranging from extremely quiet to

loud (*pppp* to *f*). The duration of one member of each sequence is 0.074 seconds, and $F_{16} = 987$ members are used of each sequence, resulting in a piece duration of 73.038 seconds. 0s in the absent residues sequence were sonified as silence, resulting in periodic gaps on the right speaker.

Figure 91: *Absent and Unique Residues no. 3.*

Collection XIV

Absent and Unique Residues no. 3
Casey Mongoven
May 18, 2012

classification of work: `audio-visual`
synthesis engine: `Csound` `show Csound orchestra`
synthesis technique: `granular synthesis`

description of sequences sonified

A066853 `number of unique residues in Fibonacci sequence mod n 987 members used`

A118965 `number of residues absent in Fibonacci sequence mod n 987 members used`

global static parameters:

sequence offset: `1`
number of channels: `2`
duration of single member of sequence: `0.074 seconds`
piece duration: `73.038 seconds`
grain: `raw charcoal 8`

individual static parameters:

location:
−30°
30°

global dynamic parameters:

A066853

integer	loudness
1	pppp
2	pppp
3	pppp
4	pppp
5	pppp
⋮	⋮
745	mf
750	mf
875	f

Figure 91: *Absent and Unique Residues no. 3* (continued).

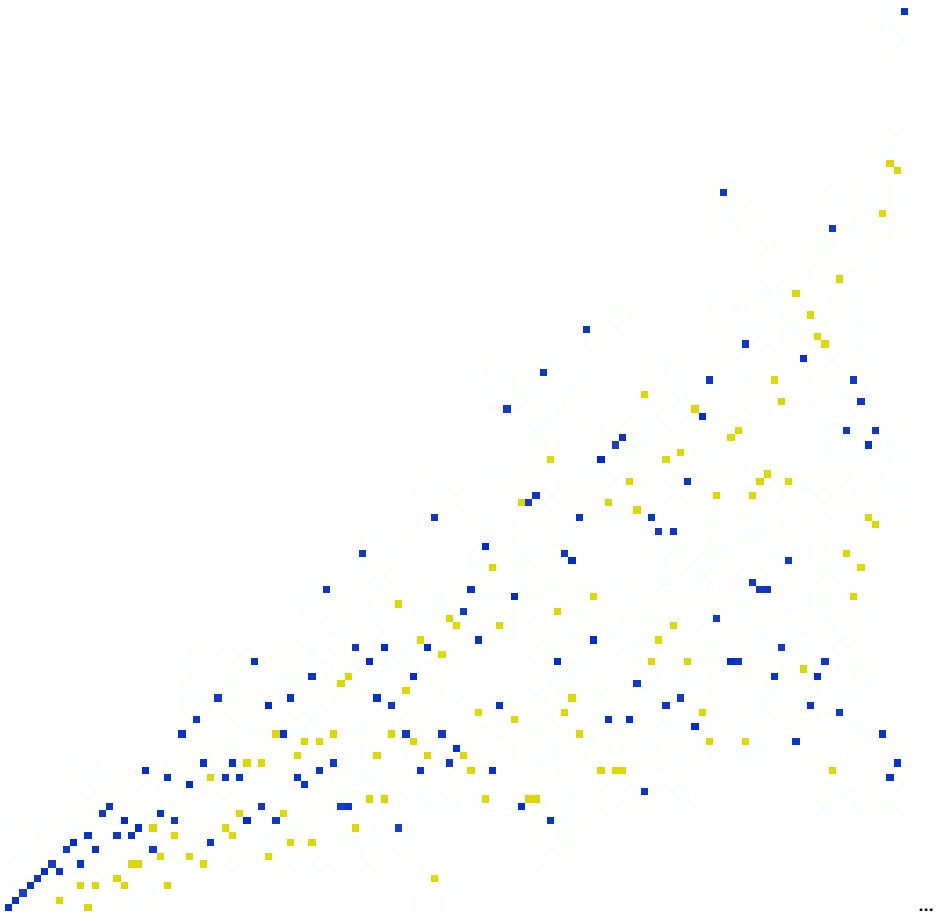
A118965

integer	loudness
1	pppp
2	pppp
4	pppp
5	pppp
7	pppp
⋮	⋮
928	ff
937	ff
966	ff

987 values used of Sloane's A066853: 1, 2, 3, 4, 5, 6, 7, 6, 9, 10, 7, 11, 9, 14, 15, 11, 13, 11, 12, 20, 9, …, 555, 149, 614, 739, 61, 745, 94, 21

987 values used of Sloane's A118965: 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 4, 1, 4, 0, 0, 5, 4, 7, 7, 0, 12, …, 425, 832, 368, 244, 923, 240, 892, 966

graph of sequences: 1) A003263, 987 values; 2) A103344, 987 values.



The graph at the end of the score had to be truncated for practical reasons. As can be seen, these sequences also display natural contrary motion when sonified together; however, the sequences' relatively erratic behavior obscures this to some degree as does the fact that only a single dynamic parameter is used in the sonification that is not frequency.

6.11 Signature Sequences

Signature sequences were briefly explained in section 5.1. If R is a positive irrational number and we arrange the set of all numbers $i+jR$ in order of increasing value, where i and j are positive integers, $i_1 + j_1R$, $i_2 + j_2R$, $i_3 + j_3R$, ..., then $< i_1, i_2, i_3, \cdots >$ is the signature sequence of R [82].

6.11.1 φ *and* Φ *Signature Sequences no. 3* (2011)

In this audio-visual work from *Collection XI* [113], two signature sequences are sonified simultaneously with a synchronized point graph.

The synthesis technique utilized in this work is based on STFT (short-time Fourier transform). Each note articulated in this piece is derived from spectral data derived from the same sample of a Daniel Gehrs Winery wine glass being struck.

As mentioned earlier, one of the challenges with signature sequences in particular, due to the relatively even distribution of the integers in such sequences, is finding a rational and suitable point of termination. Generally, a point of termination at either the highest point the sequence has reached, or at an occurrence of the integer 1, sounds best. Regardless of where one stops, the compositional result with signature sequences – and many other sequences with such uniform distribution of the integers – ends up sounding a bit as if one had suddenly torn it off at the end. In the case of this composition, 351 members of both sequences were utilized, beginning with the first members of the sequences. This resulted in a range of 1–21 for the signature sequence based on φ, and a range of 1–33 for the signature sequence based on Φ.

Each integer is attributed a duration of 0.05 seconds, resulting in a piece length of $351 * 0.05$ seconds $= 17.55$ seconds. Two dynamic parameters are used: frequency and simulated location. The latter is an individual dynamic parameter (each sequence has been given its own location), the former is global. The unit interval of the tuning used in this piece is $\varphi^7 + 1$, and the frequency orientation is descending. Dynamic level (loudness) is a static parameter: each piece is attributed the dynamic level *mezzoforte* (medium loud). Attack and release values – also static parameters of this composition – of 0.0055 and 0.0089 seconds were used for each integer.

Figure 92: *φ and Φ Signature Sequences no. 3.*

Collection XI

φ and Φ Signature Sequences no. 3
Casey Mongoven
October 29, 2011

classification of work: audio-visual
synthesis engine: Csound show Csound orchestra
synthesis technique: STFT-based phase vocoder

description of sequences sonified

A084532 Arranging the numbers s + jφ in increasing order, where s and j are positive integers, the sequence of s's is the signature sequence of φ. φ is equal to (-1 + sqrt(5))/2. 351 members used

A084531 Arranging the numbers s + jΦ in increasing order, where s and j are positive integers, the sequence of s's is the signature sequence of Φ. Φ is equal to (1 + sqrt(5))/2. 351 members used

global static parameters:

offset: 1
pitch orientation: descending
temperament: phi^7 + 1
number of channels: 2
note value: 0.05 seconds
piece length: 17.55 seconds
spectral data: gehrs glass seven
dynamic: mf
attack: .0055 seconds
release: .0089 seconds

global and individual dynamic parameters:

A084532

integer	approximate frequency Hz	simulated location degrees
1	873.0	-1.635
2	843.9	-2.394
3	815.8	-3.152
4	788.7	-3.910
5	762.4	-4.669
6	737.0	-5.427
7	712.5	-6.185
8	688.8	-6.944
⋮	⋮	⋮
19	474.6	-15.286
20	458.8	-16.044
21	443.5	-16.802

Figure 92: *φ and Φ Signature Sequences no. 3* (continued).

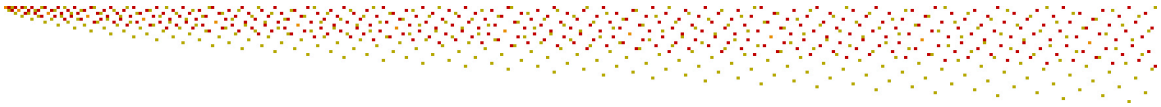
A084531

integer	approximate frequency Hz	simulated location degrees
1	873.0	25.903
2	843.9	25.144
3	815.8	24.386
4	788.7	23.628
5	762.4	22.869
6	737.0	22.111
7	712.5	21.353
8	688.8	20.594
⋮	⋮	⋮
31	316.1	3.152
32	305.6	2.394
33	295.4	1.635

351 values used of Sloane's A084532: 1, 1, 2, 1, 2, 1, 3, 2, 1, 3, 2, 4, 1, 3, 2, 4, 1, 3, 5, 2, 4, …, 14, 6, 19, 11, 3, 16, 8, 21

351 values used of Sloane's A084531: 1, 2, 1, 3, 2, 4, 1, 3, 5, 2, 4, 1, 6, 3, 5, 2, 7, 4, 1, 6, 3, …, 12, 25, 4, 17, 30, 9, 22, 1

graph of sequences: 1) A084532, 351 values; 2) A084531, 351 values.



6.12 Fractional Part of GR Multiples

We will conclude this section with another rich source for sonification – the fractional part of GR multiples. It was mentioned in section 3.6.1 that GR could produce the most uniform possible distribution of seeds on a seed head [31]. If we take a number line with the interval of [0,1), meaning that we include the point at 0 but exclude the point at 1, we can divide this number line into two line segments by placing a single point at a value determined by a real number R. For $R \geqslant 1$, we can take the fractional part of R, frac(R) and plot it on our number line. In the same way, the number line could be divided into $n + 1$ segments by plotting the values given by

$$\text{frac}(r),\ \text{frac}(2R),\ \ldots,\ \text{frac}(nR).$$

If R is a rational number, then only a limited number of distinct points occur on the number line according to the process described. For example, if $R = 0.5$, then all frac($0.5m$) $= 0$ for even m, and all frac($0.5m$) $= 0$ for odd m. The most uniform possible distribution of points in the interval [0,1) is achieved with $R = \Phi$ [26].

6.12.1 *Fractional Part of* Φ *Multiples no. 1* (2008)

Peter Anderson made me aware of some of the interesting properties of the fractional part of GR multiples at the 13th conference of the Fibonacci Association, in Patras. I began composing *Fractional Part of* Φ *Multiples no. 1* while on a bus traveling from Patras to Athens and finished the compositional process in Weimar [114]. The score is shown below:

Figure 93: *Fractional Part of* Φ *Multiples no. 1.*

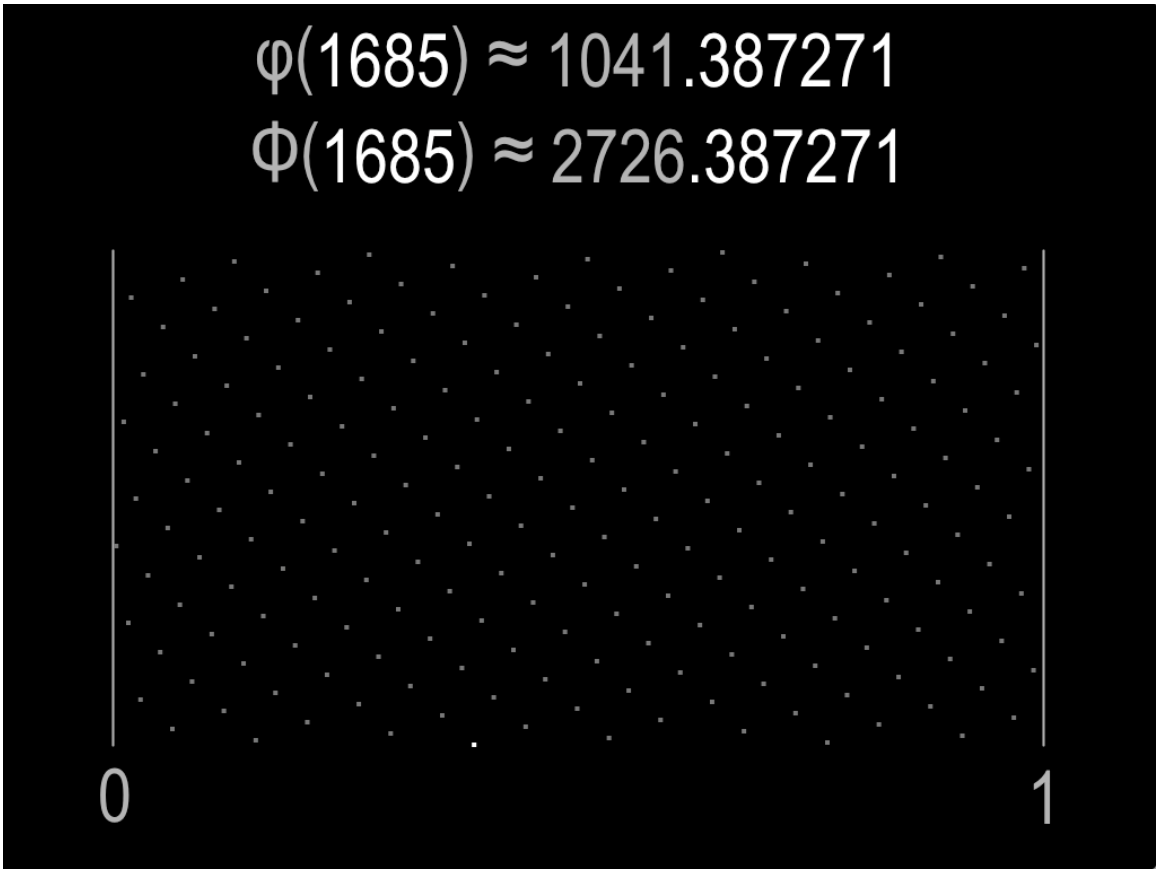
Collection VIII

Fractional Part of Φ Multiples no. 1
Casey Mongoven
July 16, 2008

description of sequence: `fractional part of Φ*n (or φ*n - same sequence) where n is an integer greater than 0`
description of piece: `The sequence is spatialized using clicks (impulses) of 1 sample. 0 indicates that the sequence value is heard on the left side; 1 indicates that the value sounds like it is coming from the right side. A value of .5 (which does not occur in the sequence) would sound as if it were coming from the middle. The direction can be reversed. A mixture of panning and delay is used to simulate spatial location. A number of samples is chosen as the duration for one click (the impulse and the silence thereafter); the number of clicks per second must be less than 877.963. The duration of the piece is not defined (see Csound file). This piece can also be realized as a continuously running installation with graphics (see catalogue).`
offset: `1`
number of members used: `undefined`
number of channels: `2`
piece length: `undefined`
note value: `at least .001139 seconds`
number of simulated spatial locations: `undefined`
synthesis technique: `impulses of 1 sample`
dynamic: `at least mp, not louder than ff`
simulated spatial location: `left to right or right to left`
delay: `-.000435 to .000435 or .000435 to -.000435`
synthesis language used: `Csound`
dedication: `to Peter Anderson`

Fractional Part of Φ *Multiples no. 1* was one of my earliest audio-visual compositions. The video component was created using Processing and is considered to be optional. The following is a still frame of the video, in which one can clearly see the uniform distribution of frac($R\Phi$):

Figure 94: Still frame from *Fractional Part of* Φ *Multiples no. 1.*



7 An Experiment in the Aesthetic Value of Sonified Mathematical Objects

Our main research question, if the reader recalls, was formulated as follows: Do sonifications of mathematical objects closely related to FS and GR hold some special aesthetic significance or advantage over less closely related sonified mathematical objects? This final section of the dissertation describes the methodology, hypotheses and results of an experiment designed to contribute to answering this question. 170 subjects were involved in the experiment. Five different pairs of sonifications were compared for their aesthetic value relative to one another. In each pair, one of the sonified mathematical objects was fundamentally related to FS and GR, whereas the other was not. For each pair, the research question is: Which sonification is found to be more aesthetically pleasing? The results for each pair are a component in answering our main research question. Ideally, we would have many more pairs, but due to the temporal nature of sonification and limited financial resources, we will make do with five contrasting pairs in this experiment.

The structure of the experiment will also allow us to pose an additional research question about the consistency of subjects choosing what sonifications they find more aesthetically pleasing. Another important research question is thus: When presented with the pairs of sonifications in this study, do subjects choose which sonifications are more aesthetically pleasing in a seemingly random fashion, or is their level of consistency greater than that which would occur purely by chance?

The experiment is unprecedented in at least four ways:

- No scientific experiment has ever been carried out in an attempt to determine the aesthetic value of FS and GR in the audio domain or music.
- No composer has ever carried out a scientific experiment on the aesthetic value of his work.
- As of yet, no other experiments are known to have been carried out in which sonifications are presented to subjects with the purpose of comparing their relative aesthetic value.
- No experiment has been carried out in an attempt to determine subjects' consistency in comparing the aesthetic value of sonifications.

We will begin with a description of the five pairs of sonifications compared in the experiment. A description of the experimental methods will then be presented. Following this, the hypotheses and results will be outlined.

7.1 The Five Pairs of Sonifications

Various types of mathematical objects were chosen for sonification in this experiment: binary sequences, real number sequences, numeral systems, integer sequences, and single irrational numbers. In choosing the Fibonacci-related mathematical objects

to be used in the experiment, I looked to my previous sonifications and attempted to select a group of five *contrasting* mathematical objects for which I could find some analog (structurally similar) mathematical object that was not closely related to FS and GR. The analog nature of the mathematical objects selected for each pair functions as a control in the experiment. In choosing the mathematical objects to be compared to those related to FS and GR, I made a conscious attempt to select those that were interesting to me personally and that I felt would make *good competition* – and those that would be significantly different sounding enough to enable subjects to make decisions about which they found more aesthetically pleasing. For each pair, the method of sonification was the same – I simply "plugged in" different numbers; the method of sonification functions as a control for each pair in the experiment. In three cases, different points of termination were chosen in the sonifications in a pair (see 5.1.3), as terminating these sonifications at the same point could have potentially created a bias for one sonification or the another.

Together, the five pairs of sonifications presented here are components of the composition I call *Experiment I Suite* [115]. All of the sonifications utilized in the experiment were created using Csound in stereo. The subjects used headphones to listen to the sonifications. All sonifications involved in the experiment and the original Csound files are available at http://caseymongoven.com/experiment/I. Listening to the sonifications will make the elliptical descriptions given in this section more clear.

7.1.1 Golden String vs. Thue-Morse Sequence

In this pair, a sonification of the golden string, discussed in 6.3, was compared to another binary sequence: the Thue-Morse sequence (A010060 in [2]).

Description of the Sequences Like the golden string, the Thue-Morse sequence can be generated in various ways. As mentioned earlier, the golden string can be generated by means of a substitution map:

$$\begin{aligned} 0 &\to 1 \\ 1 &\to 10. \end{aligned}$$

We then take the limit of the sequence that is created by starting with 0:

$$0 \to 1 \to 10 \to 101 \to 10110 \to 10110101 \to 1011010110110 \to \dots$$

and obtain $< 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, \cdots >$.

We can generate the Thue-Morse sequence by means of a substitution map in a similar fashion:

$$\begin{aligned} 0 &\to 01 \\ 1 &\to 10. \end{aligned}$$

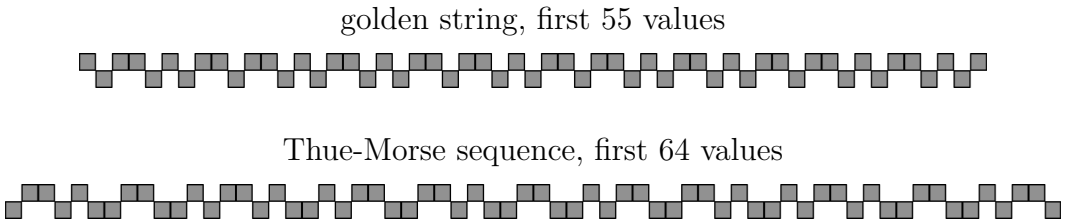
We then take the limit of the sequence created starting with 0:

$$0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \dots$$

and obtain $< 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \cdots >$ [92].

Whereas the golden string grows according to FS in the above outlined process, the Thue-Morse sequence grows according to 2^z, where $z \in \mathbb{Z}^*$.

Figure 95: Graphs of the golden string and the Thue-Morse sequence.



Description of the Sonifications The sonification of these sequences was extraordinarily simple:

- Each integer was attributed an equal duration of 0.109 seconds.
- The 1s were placed on the right speaker, the 0s on the left.
- The same sample was used for the 1s and 0s, created from the sound of a single pop (a grain) from burning charcoal passed through a resonance filter (*reson* in Csound) with a center frequency of 256 and a bandwidth of 128. In such a way, the sonification was purely spatial – no other dynamic parameters were utilized.
- $F_{13} = 233$ members of the golden string were played; the duration of the sonification was 25.397 seconds. $2^8 = 256$ members of the Thue-Morse sequence were played; the duration was 27.904 seconds.

7.1.2 Fractional Part of φ vs. $\frac{1}{8}$ Multiples

This pair consisted of a sonification of the fractional part of GR integer multiples and an analog sonification of the fractional part of $\frac{1}{8}$ integer multiples. The fractional part of GR multiples was discussed in 6.12.

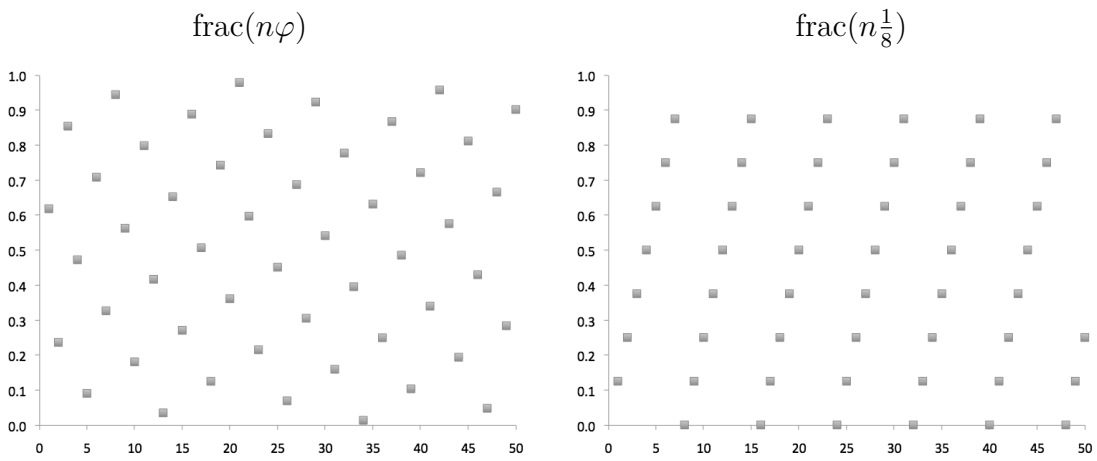
Description of the Sequences The sequence based on GR, $\text{frac}(n\varphi)$, consists of irrational numbers, whereas $\text{frac}(n\frac{1}{8})$ consists of rational numbers. The sequences begin as follows:

Figure 96: Graphs of fractional part of φ and $\frac{1}{8}$ integer multiples.

n	frac($n\varphi$)	frac($n\frac{1}{8}$)
1	.61803...	.125
2	.23607...	.25
3	.85410...	.375
4	.47214...	.5
5	.09017...	.625
6	.70820...	.75
7	.32624...	.875
8	.94427...	.0
9	.56231...	.125
10	.18034...	.25
11	.79837...	.375
↓	↓	↓

The sequence of frac($n\frac{1}{8}$) is periodic with period 8. In contrast, frac($n\varphi$) is aperiodic, as it is based on an irrational number. As mentioned in 6.12, the most uniform possible distribution of points in the interval [0,1) is achieved with frac($n\varphi$). frac($n\varphi$) is equal to frac($n\Phi$). The first 50 members of each sequence are graphed below:

Figure 97: Graphs of $n\varphi$ and $n\frac{1}{8}$.



Description of the Sonifications These sequences were also sonified exclusively in the spatial domain.

- Each number R was attributed an equal duration of 0.09375 seconds.
- A combination of amplitude panning and interaural time delay was used as a means of spatialization.
- A single impulse (of one sample) was used to represent each number.

- Numbers $R < .5$ emanated from the left side, $R = .5$ emanated exactly from the middle, and $R > .5$ emanated from the right side.
- Each sonification was played for exactly 20 seconds.

7.1.3 Dual Zeckendorf Representations vs. Binary Integers

In this pair, the dual Zeckendorf representations detailed in 6.7 were pitted against integers represented in the binary numeral system.

Description of the Numeral Systems As discussed earlier, the dual Zeckendorf representations form a system of representing integers with FS as a base in which no consecutive 0s occur in a representation. In this system, $n \in \mathbb{N}$ can be represented as

$$\sum_{j=2}^{M} c_j F_j,$$

where $c_j \in \{0, 1\}$ and $c_j + c_{j+1} \geqslant 1$.

In the binary numeral system, $n \in \mathbb{N}$ is represented as

$$\sum_{j=0}^{M} c_n 2^n,$$

where $c_n \in \{0, 1\}$.

The first 13 members of each sequence are shown in the following:

Figure 98: Dual Zeckendorf representations and the binary numeral system.

n	**dual Zeckendorf representation**	**binary numeral system**
1	1	1
2	10	10
3	11	11
4	101	100
5	110	101
6	111	110
7	1010	111
8	1011	1000
9	1101	1001
10	1110	1010
11	1111	1011
12	10101	1100
13	10110	1101
↓	↓	↓

Every single possible bit combination is exhausted in the binary numeral system, and every possible bit combination in which no consecutive 0s occur is exhausted in the dual Zeckendorf representations. A graph of these sequences is shown below, in which each black square represents a 1:

Figure 99: Graphs of dual Zeckendorf representations and binary integers.

dual Zeckendorf representations $1 \geqslant n \leqslant 143$



binary numeral system $1 \geqslant n \leqslant 128$



Description of the Sonifications The sonification of these sequences was more sophisticated than those covered up till this point. The dynamic parameters utilized were: simulated spatial location, frequency, and loudness.

- Each representation was attributed an equal duration of 0.27 seconds.
- Karplus-Strong string synthesis (*pluck* in Csound) was used.
- The tuning utilized was the series of harmonic partials. A fundamental frequency of 96 Hz was chosen. The bit numbers in each numeral system were mapped directly to the partials, beginning the bit number count at 1. As such, in both systems and both sonifications, the first bit (representing the decimal number 1 in both systems) was represented by a frequency of 96 Hz, the second bit by 2*96 Hz, the third by 3*96 Hz, the fourth by 4*96 Hz, etc.
- A combination of amplitude panning and interaural time delay was used as a means of spatialization.
- Lower bit numbers emanated from the right side, higher ones from the left.
- Lower bit numbers were quieter, higher ones were louder.
- $F_{12} - 1 = 143$ members of the dual Zeckendorf representations were played; the duration of the sonification was 38.61 seconds. $2^7 = 128$ members of the binary numeral system were played; the duration was 34.56 seconds. The number of members sonified correspond to the graphs above.

7.1.4 Signature Sequences: Φ vs. Ramanujan-Soldner Constant

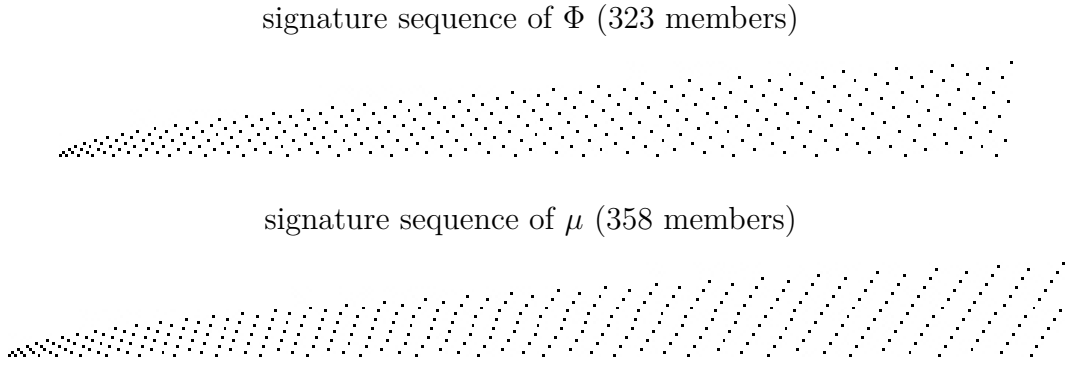
Signature sequences were discussed in 6.11. In this pair, the signature sequence based on the larger GR value Φ was pitted against another signature sequence based on the Ramanujan-Soldner constant.

Description of the Sequences The Ramanujan-Soldner constant μ is defined as the positive root of the logarithmic integral $\mathrm{li}(x)$ for real x. It is an irrational number with an approximate value of $\mu \approx 1.4513692349$ [92].

As mentioned before, given positive integers i and j, if R is a positive irrational number and we arrange the set of all numbers $i + jR$ in order of increasing value, $i_1 + j_1R, i_2 + j_2R, i_3 + j_3R, \ldots$, then $< i_1, i_2, i_3, \cdots >$ is the signature sequence of R [82].

The signature sequence of Φ (A084531 in [2]) is characterized by great uniformity in spacing, whereas the signature sequence of μ (A227539 in [2]) is less uniform in this regard. In both of the following graphs, the lowest integers occurring are 1 and highest 32:

Figure 100: Signature Sequences: Φ and the Ramanujan-Soldner Constant.



Description of the Sonifications The dynamic parameters utilized in these sonifications were: simulated spatial location, frequency, and loudness.

- Each representation was attributed an equal duration of 0.068 seconds.
- A form of wavetable synthesis was used in which the wavetable is resynthesized according to the frequency utilized for an integer using FFT in order to avoid aliasing. The wavetable used was created with a viola.
- The tuning utilized was the series of harmonic partials. A fundamental frequency of 94 Hz was chosen. The integers in the sequence were mapped directly to the partial numbers, so that in both sonifications, the integer 1 was represented by 94 Hz, 2 by 2*94 Hz, 3 by 3*94 Hz, 4 by 4*94 Hz, etc.

- A combination of amplitude panning and interaural time delay was used as a means of spatialization.
- Lower integers emanated from the left side, higher ones from the right.
- Lower integers were quieter, higher ones were louder.
- 323 members of the signature sequence of Φ were played; the duration of the sonification was 21.964 seconds. 358 members of the signature sequence of μ were played; the duration was 24.344 seconds. The number of members sonified corresponds to the graphs above.

7.1.5 Φ vs. $\sqrt{3}$

Φ was discussed in 1.1. In this pair, Φ was pitted against another irrational number: $\sqrt{3}$. This is the only pair in the experiment in which single numbers were sonified.

Description of the Numbers As continued fractions, Φ and $\sqrt{3}$ can be expressed as

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}} \quad \text{and} \quad 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\ddots}}}}},$$

respectively [116]. $\Phi \approx 1.6180339888$ and $\sqrt{3} \approx 1.7320508076$.

Description of the Sonifications The sonification of these two numbers was extremely simple. Steady pulses were used with a tempo ratio between the speakers of 1:Φ for one sonification, and 1:$\sqrt{3}$ for the other.

- The left speaker in each sonification held a steady pulse of 0.32 seconds. The right speaker held a steady pulse of $0.32 \times \Phi \approx 0.517770876$ seconds for one sonification, and $0.32 \times \sqrt{3} \approx 0.554256258$ seconds for the other.
- Both speakers started simultaneously.
- The same sample of a single strike of a drum stick on a Remo drum pad was used on both speakers at the same amplitude.
- Each sonification was played for exactly 21 seconds.

7.1.6 Synopsis

The analog nature of the mathematical objects paired in each case and the method of sonification for each pair being identical functioned as a control. The lengths of the sonifications were chosen so that sometimes the Fibonacci-related sonifications were longer in duration, and sometimes vice versa. The following table shows the approximate durations of the sonifications in seconds along with their combined durations:

Figure 101: Approximate durations of the sonifications.

mathematical objects	**FS-related**	**not closely related to FS**
binary sequences	25.397s	27.904s
fractional part of integer multiples	20s	20s
numeral systems	38.61s	34.56s
signature sequences	21.964s	24.344s
irrational numbers	21s	21s
total	126.971s	127.808s

As can be seen, there is difference of less than one second – less than 0.7% – in the total duration for each class of sonification. The sonifications not closely related to FS had a slightly longer total duration.

Various synthesis techniques and dynamic parameters were used in the sonifications. The following table gives a general description of the synthesis techniques and the dynamic parameters utilized in each pair:

Figure 102: Synthesis techniques and dynamic parameters.

mathematical objects	**synthesis technique**	**dynamic parameters**
binary sequences	granular synthesis	• spatial location
fractional part of integer multiples	single-sample impulses	• simulated spatial location
numeral systems	Karplus-Strong string synthesis	• simulated spatial location • loudness • frequency
signature sequences	wavetable synthesis	• simulated spatial location • loudness • frequency
irrational numbers	synthesis with raw samples	• tempo ratio between two speakers

7.2 A Description of the Experimental Method

This experiment was carried out at UCSB from January 15, 2013 to April 1, 2013. The Media Arts and Technology Department (henceforth MAT) provided financial means for the experiment.

7.2.1 The Location

Two separate rooms at UCSB were utilized in the experiment, both located in Phelps Hall: Phelps 3312 and Phelps 3314 (henceforth 3312 and 3314). In order to enter 3312, one had to go through 3314, as these rooms were connected by a single door. Generally, the door to room 3314 remained closed as did the door connecting 3314 and 3312. The following pictures depict these two rooms at the time of the experiment. The dimensions of each room is given below the pictures:

Figure 103: The location.



Phelps 3314: 153 in. × 119.5 in. × 98.5 in.



Phelps 3312: 153 in. × 114.5 in. × 98.5 in.

In order to avoid social facilitation or influence, the experimenter was generally not present in the rooms during the experiment, but performed routine checks in both rooms to see that the subjects were on task. For 3314, this could be accomplished without entering the room. It was necessary to enter 3314 to check on subjects in 3312, as they are connected by a door, as mentioned. In every case, the experimenter attempted to make these checks as non-invasive and unnoticeable as possible. Ideally, the experimenter would have had access to two, separate, perfectly acoustically isolated rooms with some remote form of surveillance in order to mitigate extraneous variables.

In order to ensure that no other persons disturbed the experiment in progress, a sign was placed on the door with instructions not to knock, which all subjects followed. Subjects were also routinely asked to exit the rooms quietly in order not to disturb any other test in progress.

7.2.2 The Equipment

Most of the equipment used in the experiment was provided by MAT. The following picture depicts the equipment used in 3314:

Figure 104: The equipment.



The following is a list of the equipment used:

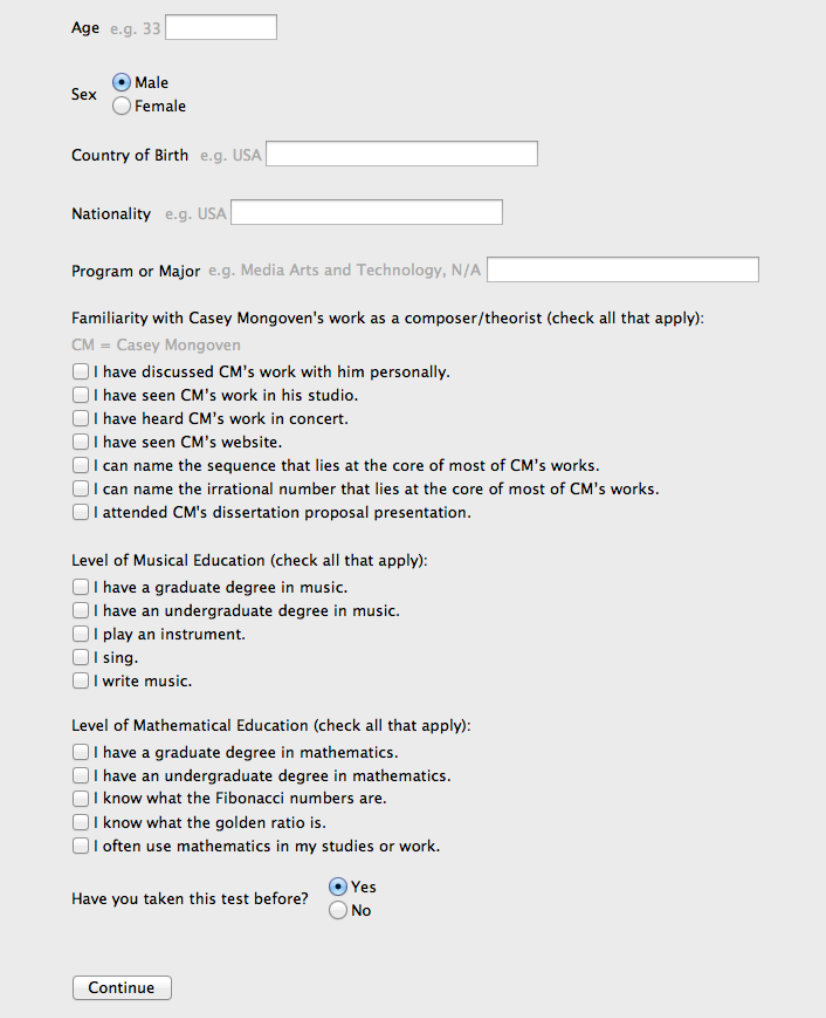
- 2 Logitec K120 keyboards
- 2 Logitec M110 mouses
- 1 Dell 1908FPt monitor
- 1 Samsung SyncMaster 191N
- 2 Late 2012 Mac minis
  - Processor: 2.5 GHz Intel Core i5
  - Memory: 4 GB 1600 MHz DDR3
  - Graphics Card: Intel HD Graphics 4000 512 MB
  - Operating System: OS X 10.8.3 (12D78)
- 2 Sennheiser HD239 headphones

7.2.3 The Program and Testing Methodology

The experimenter designed a program specifically for use in this experiment, written in Cocoa (Objective-C) in Xcode. The program was rigorously tested before deployment to ensure accurate results.

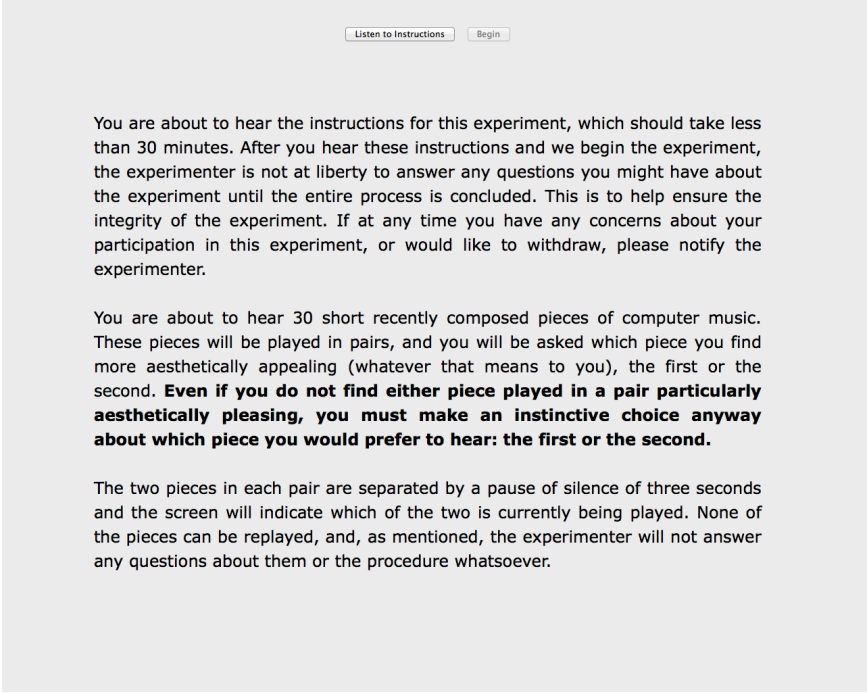
The first component of the program was a survey designed to gather information pertaining to the subjects: their age, sex, country of birth, nationality, program or major, familiarity with the experimenter's work, level of musical education, level of mathematical education, whether the subject had taken part in the experiment before. The following screenshot of the program depicts the survey:

Figure 105: The survey.



Following the survey, the program presented instructions for the experiment, requiring the subjects to listen to the information as it was presented in text form on the screen:

Figure 106: The instructions.



In addition to these written instructions, the recorded instructions asked that the subjects ensure that they were wearing the headphones in the proper orientation, and included a test of the right and left speakers in order to aid them in this task.

For the testing portion of the program, each of the five pairs of sonifications was repeated three times, making for a total of 15 pairs presented. The order in which the pairs were presented was pseudo-random, according to the Fisher-Yates shuffle [117], seeded from the system clock. The order of the sonifications within each pair was also pseudo-random, according to the Objective-C function arc4random(), which is seeded automatically upon initialization. In such a way, each time the same pair was presented, approximately 50% of the time it happened that the order of the individual sonifications was switched. As mentioned in the instructions, for the sake of clarity, the screen indicated whether the first or the second sonification was currently being played. There was a pause of approximately five seconds between sonifications in each pair. The program allowed subjects to take as long as they needed to determine which sonification they found more aesthetically pleasing. The program required subjects to click on one of two buttons indicating which sonification they found more aesthetically appealing. After a sonification was chosen, there was a pause of approximately five seconds before the next sonification began. The following is a screenshot from the testing portion of the program:

Figure 107: The testing portion.



Upon completion of a test, the program created a .txt file containing the information from the survey and the selected sonifications. The file was titled with a time-stamp, indicating the time at which the subject completed the experiment.

No subject was allowed to take part in the experiment more than once. When subjects arrived, they were required to fill out a consent form and given oral instructions about the task required in the experiment in order to reinforce the instructions provided by the program itself. The subjects were not told that each pair would be presented three times, nor that there were five pairs. On several occasions, the results of a test were discussed with subjects with whom the experimenter was personally familiar (non-volunteers) under the pretext that the subjects would not divulge any information shared. Subjects who were not volunteers were paid upon completion of the experiment and signed a form indicating that they had received compensation.

As a control, the experimenter performed routine checks to ensure that the volume of both Mac minis remained exactly the same throughout the course of the experiment. The volume was set so that – according to the experimenter – none of the sonifications were too loud or too quiet. At no point was the volume found to have been changed.

7.2.4 The Database and Experiment Log

Upon completion of the experiment, a MySQL database was set up and a C++ program was created to parse the information contained in the .txt files created with the program. Some small changes that had to be made to the demographic information in the raw .txt files are detailed in Appendix C. The database also contains information

from a detailed log the experimenter kept on his experience with each participant and information about routine procedures being carried out. Comments about each subject are included as well as information about whether the experimenter detected disturbances during the course of the experiment. Upon completion of the database, a webpage was created where visitors can view and filter the results as well as obtain more information about the experiment: http://caseymongoven.com/experiment/I.

7.2.5 The Subjects

The majority of the subjects were UCSB undergraduate students. 140 of the 170 subjects were paid for their participation, recruited through UCSB's Undergraduate Research Participation System (SONA). The remaining 30 were volunteers, mostly from MAT. None of the paid subjects were known to the experimenter, whereas all of the volunteers were. The first 100 paid subjects received six dollars for their participation, whereas the remaining 40 received five dollars due to budget restrictions. High interest in paid participation in the experiment was evident: 49 subjects signed up on SONA for participation in the first week within less than 15 hours. Subjects were assigned a subject number (1–170) according to the order in which they completed the test.

The following is a list of the five paid subjects whose results were discarded for various infractions:

20. The subject was observed talking to himself while listening to a sonification.

36. The subject was observed removing the headphones during the experiment.

53. The subject implied he had removed the headphones during the experiment.

68. The subject spoke when distracted by the experimenter during a routine check.

99. The subject was twice observed using a cell phone during the experiment.

The following is a summary of some of the demographic information gathered from the survey of the 165 subjects whose results were not discarded:

Figure 108: Summary of the demographics of the samples.

age	mean: 23.3, median: 20, mode: 19 youngest: 18, oldest: 67
sex	♀ 63.6%, ♂ 36.4%
number of unique majors/programs **most common majors/programs**	50 Media Arts & Technology: 23 Psychology: 16
unique countries of birth **most common countries of birth**	20 USA: 132, China: 7
unique nationalities **most common nationalities**	15 USA: 147, Canada: 3

Detailed information on demographic information is available on the webpage dedicated to the experiment.

7.3 Hypotheses and Results

In this section, we outline the hypotheses for the various research questions in this dissertation and present results. In order to adequately address our research problems, we will approach the analysis of the data from several different angles. Because this research is unprecedented, it will be necessary to introduce several new concepts and terms in the course of stating and testing our hypotheses.

Because each pair of sonifications is presented three times in the experiment, there are two primary approaches to analyzing the data gathered. First, we can look at the data irrespective of a subject's consistency in choosing which sonification is most aesthetically pleasing in a pair. With this approach, if a subject chooses a Fibonacci-related sonification two times for a pair and the other sonification once, for example, we can view the subject as leaning towards the Fibonacci-related sonification. Multiple subjects stated that they became aware that the pairs were being repeated, and a few said that they had changed their minds about which sonification in a pair they found most aesthetically pleasing. This is not dissimilar to the way that Fechner allowed subjects to choose several different rectangles if they could not decide on one in particular. The difference here is that there are only two sonifications to choose from. The second approach we can take is to simply discard any results where a subject's choice was not consistent. We will utilize both of these approaches in the analysis of our data and see whether we achieve similar results with both. A table of the most important raw data gathered in the experiment can be seen in Appendix D.

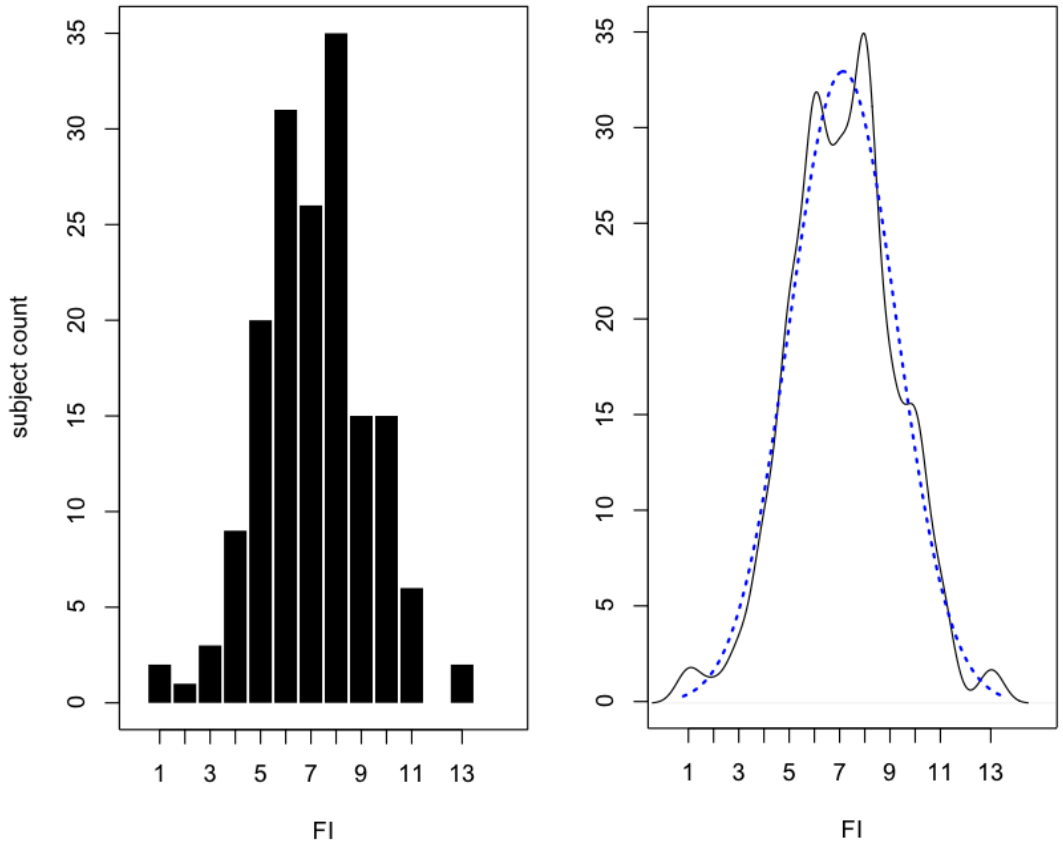
7.3.1 Hypotheses and Results for the Main Research Problem

We are studying the impact of the use of Fibonacci-related mathematical objects in sonification on the aesthetic pleasantness of the resulting sonification. The method of the creation of our stimuli is our independent variable; aesthetic preference in two-alternative forced-choice (2AFC) tests is our dependent variable.

Hypotheses for the Fibonacci Index We will call the number of times a Fibonacci-related sonification is chosen by a subject the *Fibonacci index* (henceforth FI). FI disregards a subject's consistency within individual pairs. The maximum value for FI is $15 = 5 \times 3$ and the minimum 0, as five pairs are presented three times each.

Let us first take a look at the distribution of FI for the sample of 165 subjects tested. We will use a standard bar chart and a density plot with a superimposed normal curve with the same standard deviation and mean as our sample (shown in blue) in order to determine what type of statistical test to use:

Figure 109: Distribution of the Fibonacci index.



The distribution of FI looks approximately normal, and we observe a mean value of 7.12. We can use the Shapiro-Wilk normality test to test the null hypothesis of whether the sample came from a normally distributed population [118]. We will use a *level of significance* of $\alpha = .05$ for the statistical tests in this dissertation. The level of significance is the probability of rejecting the null hypothesis when it is in fact true. In this case, $\alpha = .05$ would mean we have a 5% chance of rejecting the null hypothesis that the sample came from a normally distributed population when it in fact did.[11] The *p-value* of the Shapiro-Wilk test of our sample is .005.[12] The p-value, which we will call p, indicates the probability of obtaining a test statistic at least as extreme as the one that was actually observed strictly by chance. Because p is less than our level of significance, $.005 < \alpha$, we can assume that our population is in fact not normally distributed, which means we require a nonparametric test (one that does not assume a normal distribution).

We will use the original single sample Wilcoxon signed-rank test to test our hypothesis [120].[13] This test is designed to test a hypothesis about the median (or

[11] A significance level of .05 is an arbitrary value. It is, however, common in statistical tests in the fields of behavioral science and psychology [119].

[12] In this dissertation, p-values are rounded to three decimal places.

[13] All statistical tests in this dissertation were carried out using R, as were all graphs in this

mean) of a population distribution. We make the following assumptions in conducting a Wilcoxon signed-rank test:

- The sample has been randomly selected from the population it represents.
- The form of the values obtained in the test is that of interval/ratio data.
- The underlying population has a symmetric distribution. Under this assumption, the median equals the mean.

The version of the test we will use works as follows:

Let N be the sample size. For $i = 1, \ldots, N$, let x_i represent our sample data points and μ our hypothesized median.

1. For $i = 1, \ldots, N$, calculate $|x_i - \mu|$ (the absolute value of $x_i - \mu$).
2. Exclude x_i where $|x_i - \mu| = 0$ and reduce the sample size accordingly. Let N_r be the reduced sample size.
3. Order the remaining N_r data points from smallest to largest absolute difference, $|x_i - \mu|$.
4. Rank the data points x_i according to $|x_i - \mu|$, starting with the smallest as 1. Ties (two or more data points with the same absolute difference) receive a rank equal to the mean of the ranks they span. Let R_i denote the rank.
5. Let sgn() be the sign function. The test statistic W is then calculated as the minimum of the positive and negative rank sums:

   $W_+ = \sum_{i=1}^{N_r} [R_i \mid \operatorname{sgn}(\mu - x_i) = 1]$

   $W_- = \sum_{i=1}^{N_r} [R_i \mid \operatorname{sgn}(\mu - x_i) = -1]$

   $W = \min(W_+, W_-)$
6. The p-value can then either be approximated with a normal distribution [121], or an exact p-value can be calculated utilizing the Streiberg-Röhmel shift algorithm [122]. These methods will not be covered here, but we will use the exact p-value in our tests.

As mentioned, we observed a mean value of 7.12 for our sample for FI. We would expect a mean of 7.5 if the use of Fibonacci-related mathematical objects in sonification has no impact on aesthetic value. At first glance, it appears that the use of Fibonacci-related mathematical objects could in fact have a negative aesthetic impact.

analysis section.

We will assume for our population of UCSB students, faculty and staff, that FI has a mean value $\mu = 15/2 = 7.5$:

Null hypothesis H_0: $\mu = 7.5$ (The use of Fibonacci-related mathematical objects in sonification has no impact on the aesthetic pleasantness of the resulting sonification.)

Alternative hypothesis H_a: $\mu \neq 7.5$ (The use of Fibonacci-related mathematical objects in sonification has either a positive or a negative impact on the aesthetic pleasantness of the resulting sonification.)

We will again use a level of significance of $\alpha = .05$. In this case, this means we have a 5% chance of claiming that the use of Fibonacci-related mathematical objects in sonification has some impact on the aesthetic pleasantness of the resulting sonification when it in fact does not. H_a is considered to be a two-tailed hypothesis, because the use of Fibonacci-related mathematics in sonification could have either a positive *or* a negative effect.

We begin by creating a table containing the data required for the test, as described above. We will truncate the table for practical purposes:

Figure 110: Data for the Wilcoxon test for the Fibonacci index.

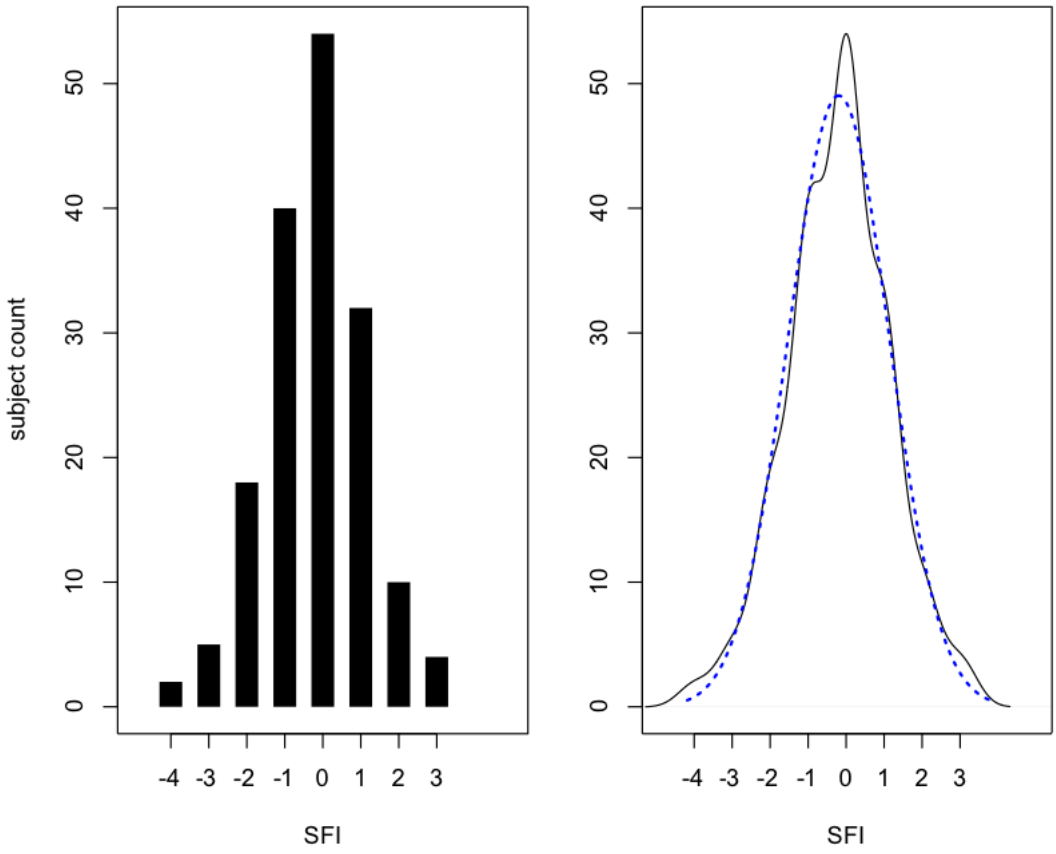
i	x_i	$\text{sgn}(\mu - x_i)$	$\lvert\mu - x_i\rvert$	R_i	$\text{sgn}(\mu - x_i) \times R_i$
8	7	-1	0.5	31	-31
14	7	-1	0.5	31	-31
16	7	-1	0.5	31	-31
20	7	-1	0.5	31	-31
22	7	-1	0.5	31	-31
⋮	⋮	⋮	⋮	⋮	⋮
12	3	-1	4.5	159	-159
39	3	-1	4.5	159	-159
101	3	-1	4.5	159	-159
19	2	-1	5.5	162	-162
87	13	1	5.5	162	162
105	13	1	5.5	162	162
129	1	-1	6.5	164.5	-164.5
133	1	-1	6.5	164.5	-164.5

We find that $W = 5451.5$, and the exact p-value is $p = .021$. We now need to see if our p-value is less than our level of significance $\alpha = .05$, and $.021 < \alpha$, so we reject the null hypothesis, H_0: $\mu = 7.5$. The use of Fibonacci-related mathematical objects in sonification appears to have a negative impact on the aesthetic pleasantness of the resulting sonification for our population.

Hypotheses for the Secondary Fibonacci Index For the *secondary Fibonacci index* (henceforth SFI), we only take into account the pairs for which a subject was consistent in choosing (i.e. those pairs where the subject chose the same sonification each of the three times it was presented). SFI is calculated as the number of pairs for which a subject consistently chose the Fibonacci-related sonification minus the number of pairs for which a subject consistently chose the sonification not closely related to FS. For example, if a subject chose one Fibonacci related pair and two pairs not closely related to FS, then SFI $= 1 - 2 = -1$. SFI ranges from -5 to 5.

Let us again look at the distribution using a bar chart and a density plot with a superimposed normal curve with the same mean and standard deviation as our sample:

Figure 111: Distribution of the secondary Fibonacci index.



We can see that the distribution of SFI is similar to that of FI. The Shapiro-Wilk test again indicates that SFI does not have a normal distribution.

We observe a mean value of $-.21$. We would expect a mean value of 0 if the use of Fibonacci-related mathematical objects has no impact on aesthetic value. For testing purposes, we assume that mean of SFI for the entire population is 0 and the use of Fibonacci-related mathematical objects in sonification has no effect, formulating our

hypotheses as follows:

Null hypothesis H_0: $\mu = 0$

Alternative hypothesis H_a: $\mu \neq 0$

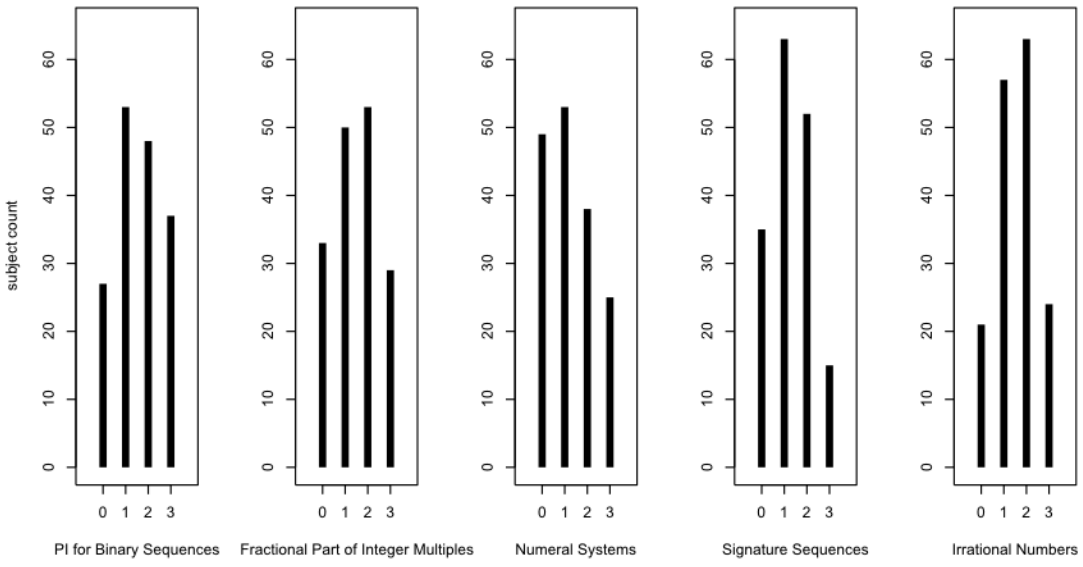
We find that $W = 2453$, and the exact p-value is .045. So we reject the null hypothesis, H_0: $\mu = 0$, because $.045 < \alpha$. We have thus obtained similar results as for FI. Again, the use of Fibonacci-related mathematical objects in sonification appears to have a negative impact on the aesthetic pleasantness of the resulting sonification for our population.

7.3.2 Hypotheses for Each Individual Pair

We now turn our attention to each individual pair of sonifications in this study and state our hypotheses. We will call the number of times a Fibonacci-related sonification is chosen in a single pair by a subject the *pair index* (henceforth PI). PI has a maximum value of 3 and a minimum of 0, as each pair is repeated three times. The *secondary pair index* (henceforth SPI) is 0 if a subject was inconsistent on a pair, 1 if a subject chose the Fibonacci-related sonification consistently, and -1 if the subject chose the sonification not closely related to FS consistently.

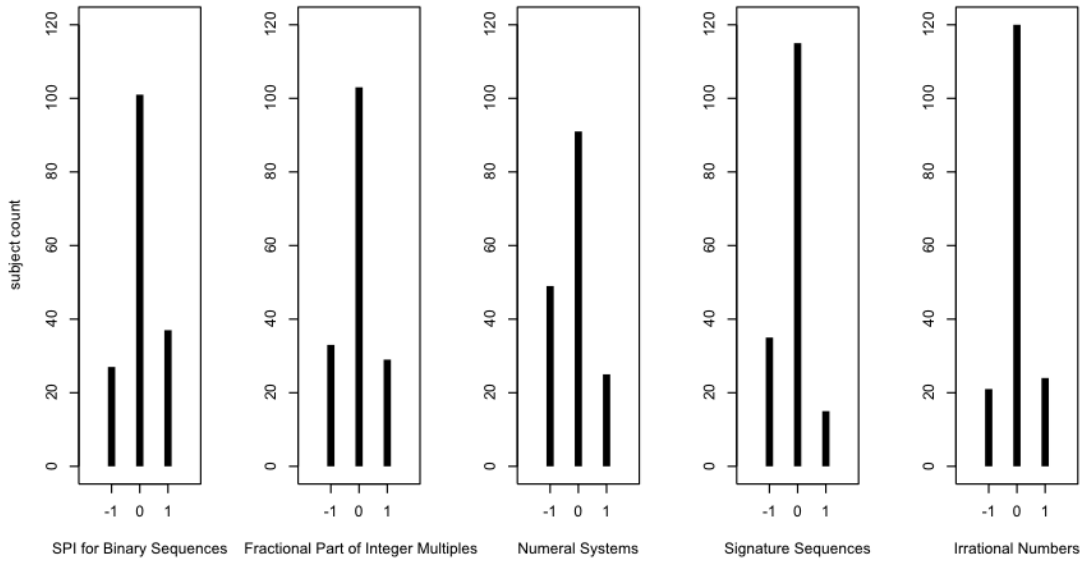
The following graphs show the distribution of PI for each pair:

Figure 112: Distribution of the pair index for each pair.



The distributions of SPI for each pair appear as follows:

Figure 113: Distribution of the secondary pair index for each pair.



In all cases, the Shapiro-Wilk test indicates a population that is non-normally distributed. We will again utilize the single sample Wilcoxon signed-rank test.

Binary Sequences We first look at PI and observe a mean value of 1.58 for this pair. We would expect a mean value of 1.5 if the use of Fibonacci-related mathematical objects has no impact on aesthetic value. For the purposes of testing, we assume that the mean of PI for the population is 1.5, i.e. the golden string and the Thue-Morse sequences are equally aesthetically pleasant when sonified in an identical manner as outlined in this dissertation.

Null hypothesis H_0: $\mu = 1.5$

Alternative hypothesis H_a: $\mu \neq 1.5$

For our PI values for binary sequences, $W = 6307.5$ and $p = .366$. We fail to reject the null hypothesis, H_0: $\mu = 1.5$, as $.366 > \alpha$.

We turn to SPI and observe a mean value of .06. We would expect a mean value of 0 if the use of Fibonacci-related mathematical objects has no impact on aesthetic value. We assume the mean for our population to be equal to 0.

Null hypothesis H_0: $\mu = 0$

Alternative hypothesis H_a: $\mu \neq 0$

In this case, $W = 877.5$ and $p = .26$. We achieve similar results as for PI and fail to reject the null hypothesis, because $.26 > \alpha$.

Fractional Part of Integer Multiples We observe a mean value for PI of 1.47. We assume that the mean of PI for the population is 1.5, i.e. the fractional part of integer multiples of GR is equally aesthetically pleasing as the fractional part of integer multiples of $\frac{1}{8}$ when sonified in an identical manner as outlined in this dissertation.

Null hypothesis H_0: $\mu = 1.5$

Alternative hypothesis H_a: $\mu \neq 1.5$

For our PI values for this pair, $W = 6656.5$ and $p = .753$. We fail to reject the null hypothesis, H_0: $\mu = 1.5$, as $.753 > \alpha$.

We again turn to SPI and observe a mean value of $-.02$. We assume the mean of SPI to be equal to 0 for our population.

Null hypothesis H_0: $\mu = 0$

Alternative hypothesis H_a: $\mu \neq 0$

We have $W = 913.5$ and $p = .704$. We achieve similar results as for PI and fail to reject the null hypothesis, as $.704 > \alpha$.

Numeral Systems We observe a mean value for PI of 1.24. We assume that the mean of PI for the population is 1.5, i.e. the sequence of dual Zeckendorf representations is equally aesthetically pleasing as the sequence of standard binary representations of integers when sonified in an identical manner as outlined in this dissertation.

Null hypothesis H_0: $\mu = 1.5$

Alternative hypothesis H_a: $\mu \neq 1.5$

For this pair, we have $W = 4960.5$ and $p = .001$. We reject the null hypothesis, as $.001 < \alpha$.

We observe a mean value for SPI of $-.15$. We assume the mean of SPI to be equal to 0 for our population.

Null hypothesis H_0: $\mu = 0$

Alternative hypothesis H_a: $\mu \neq 0$

Here, $W = 937.5$ and $p = .007$. We achieve similar results as for PI and reject the null hypothesis, as $.007 < \alpha$. With the utilized method of sonification, it appears that the sonification of the binary numeral system is more aesthetically pleasing than the analog sonification of dual Zeckendorf representations for our population.

Signature Sequences We observe a mean value for PI of 1.28. We assume that the mean of PI for the population is 1.5, i.e. the signature sequence of Φ is equally aesthetically pleasing as the sonification of the Ramanujan-Soldner constant when

sonified in an identical manner as outlined in this dissertation.

Null hypothesis H_0: $\mu = 1.5$

Alternative hypothesis H_a: $\mu \neq 1.5$

We have $W = 5123.5$ and $p = .003$. We reject the null hypothesis, as $.003 < \alpha$.

We observe a mean value for SPI of $-.12$. We again assume the mean of SPI to be equal to 0 for our population.

Null hypothesis H_0: $\mu = 0$

Alternative hypothesis H_a: $\mu \neq 0$

For the SPI of this pair, $W = 382.5$, and $p = .007$. We achieve similar results as for PI and reject the null hypothesis: $.007 < \alpha$. The sonification carried out for this study of the Ramanujan-Soldner constant appears to be more aesthetically pleasing than the analog sonification of Φ for our population.

Irrational Numbers We observe a mean value for PI of 1.55. We assume that the mean of PI for the population is 1.5, i.e. Φ is equally aesthetically pleasing as $\sqrt{3}$ when sonified in an identical manner as outlined in this dissertation.

Null hypothesis H_0: $\mu = 1.5$

Alternative hypothesis H_a: $\mu \neq 1.5$

We have $W = 6451.5$ and $p = .505$. We fail to reject the null hypothesis, as $.505 > \alpha$.

We observe a mean value for SPI of .02. We assume the mean of SPI to be equal to 0 for our population.

Null hypothesis H_0: $\mu = 0$

Alternative hypothesis H_a: $\mu \neq 0$

In this case, $W = 483$ and $p = .766$. We achieve similar results as for PI and again fail to reject the null hypothesis, as $.766 > \alpha$.

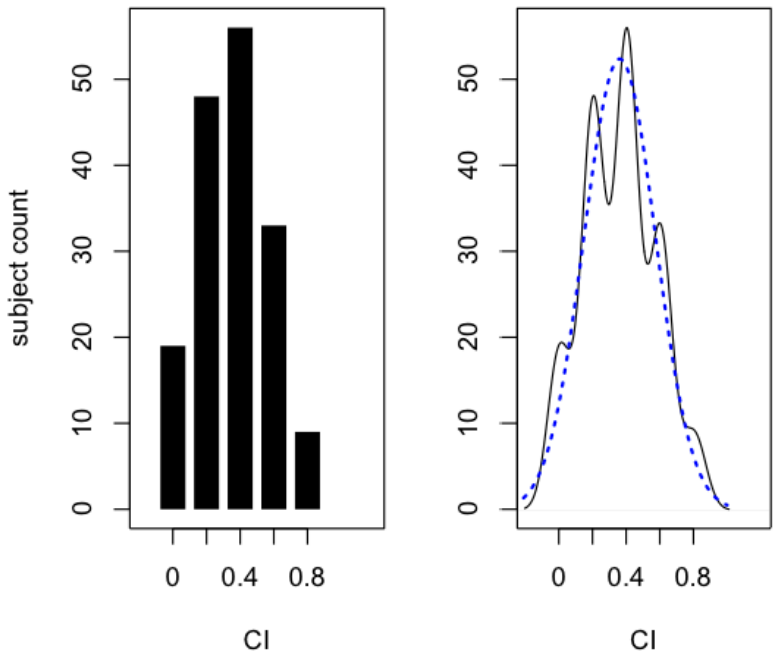
7.3.3 Hypotheses for the Consistency of the Subjects' Choices

Finally, we turn to the subjects' consistency in choosing which sonification they find most aesthetically pleasing in a pair. We will assume that the subjects desire to comply with the instructions given to them in the experiment.

The *consistency index* (henceforth CI) is given in percent and represents for how many pairs a subject chose the same sonification every time. Because there are five pairs of sonifications, CI is always either 0%, 20%, 40%, 60%, 80%, or 100% for an individual subject.

We observe a mean value for CI in our sample of 35.8%. Let us look at the distribution for CI, again using a bar chart and a density plot with a superimposed normal curve:

Figure 114: Distribution of the consistency index.



The Shapiro-Wilk test indicates a non-normally distributed population. We will once again utilize the single sample Wilcoxon signed-rank test to test our final hypotheses.

For any individual pair, the probability of being consistent in choice purely by chance is 25%, as there are $2^3 = 8$ distinct possibilities, two of which are consistent.[14] We thus assume for our population that the subjects' choices are equivalent to random choices, so that the mean of CI for the population is 25%.

Null hypothesis H_0: $\mu = .25$

Alternative hypothesis H_a: $\mu \neq .25$

H_a is again a two-tailed hypothesis: subjects are perhaps more consistent than we assume *or* perhaps they nearly always change their minds about which sonification they find more aesthetically pleasing.

For this test, we have $W = 190$ and $p \approx 0$ (2.98e−9). Thus we reject H_0. Subjects appear to be significantly more consistent in their choices of which sonification in a pair is more aesthetically pleasing than would occur purely by chance.[15]

[14] Consider for example the following bit sequences, two of which are consistent (shown in bold print): **000**, 001, 010, 011, 100, 101, 110, **111**.

[15] While a rate of consistency of 35.8% might not seem very high, a significant number of subjects mentioned to the experimenter that the sonifications sounded very similar to one another – sometimes to the point that they seemed indistinguishable.

7.4 A Summary of the Results Obtained

We carried out 13 Wilcoxon signed-rank tests and obtained significant results in seven cases.[16] In all six cases having to do with the aesthetic value of FS and GR (the seventh case had to do with subjects' consistency), results suggested that sonifications of mathematical objects related FS and GR were less aesthetically pleasing than the mathematical objects they were pitted against: there were no contradictory results in this regard.

Based on our study sample, we can summarize the results obtained for our population of UCSB students, faculty and staff, as follows:

- According to the sonifications created for this study, the use of Fibonacci-related mathematical objects in sonification appears to have a negative impact on the aesthetic pleasantness of the resulting sonification.
- It appears that the sonification carried out for this study of the binary numeral system is more aesthetically pleasing than the analog sonification of dual Zeckendorf representations.
- The sonification carried out for this study of the signature sequence of the Ramanujan-Soldner constant appears to be more aesthetically pleasing than the analog sonification of Φ.
- When presented with the pairs of sonifications in this study, subjects appear to be more consistent in their choices of which sonification is more aesthetically pleasing than would occur purely by chance.

7.5 Interpretation and Meaning of the Results

Many who were familiar with my artistic work assumed that I was going into this experiment in an attempt to provide evidence for the aesthetic value of FS and GR in music. These were generally people who were only familiar with me personally on a more superficial level. I have never once claimed that FS and GR hold a special aesthetic significance in music, only that they are a fruitful source for creating diverse and rich musical structures. It would have gone against the most basic scientific principles for me to be biased in my opinion about the research question going into the experiment. This is not the way I think as a composer or in life in general – I do not believe anything or pretend to know anything that evidence is not provided for.

We might speculate the following concerning the results:

- Concerning the reason why the evidence gathered suggests that FS and GR are aesthetically disadvantageous in sonification, this could be an artifact of the particular pairs of sonifications chosen. Every effort was made to choose

[16] Originally, two-tailed z-tests were used to analyze the data (because the distributions appear to be approximately normal). Similar results were obtained with the z-tests as with the Wilcoxon signed-rank tests.

five contrasting pairs, and sonifications within each pair were carried out in an identical manner – but five pairs is still not a large number. More tests would be useful in this regard. Another explanation could be that the unique type of symmetry and uniformity created by GR comes across in general as boring in the audio domain.

- As to why the sonification of the binary numeral system was found to be more aesthetically pleasing than dual Zeckendorf representations, I theorize it is because of the regularity and more clearly discernable symmetry of powers of 2. The ubiquitousness of the 4/4 time signature and eight-measure periods in various musical styles provide evidence for this assertion.

- As for the signature sequences, I suspect that the sonification of the Ramanujan-Soldner constant was found to be more aesthetically pleasing because its pattern was less uniform than that of Φ. If one looks at the graph provided in 7.1.4, the upward sweeps are more pronounced than for Φ.

- We failed to reject the null hypothesis for three of the individual pairs (using a total of six Wilcoxon tests). There are only two possibilities concerning these pairs: either we did not have a large enough sample or there truly is no discernable difference in aesthetic value between them. The former is far more likely, as based on our method of measurement of aesthetic value as defined by Fechner (polling), it is extremely unlikely that there would be exactly as many votes for one as for the other. It is also of interest that the three pairs for which we did not obtain statistically significant results did not use frequency as a dynamic parameter. Perhaps frequency makes it easier to distinguish between the sonifications in a pair.

- The subjects' consistency of 35.8% can be interpreted as a reflection of the subjects' general desire to comply with the instructions. It is also a reflection of the discernability of the sonifications in individual pairs. Many subjects stated that, for certain pairs, they were unable to distinguish between the sonifications.

The results obtained in this experiment do not negatively effect my opinion of the richness of FS and GR as a resource for creating unique and unified musical structures. They do, however, inform me of the lack of general aesthetic advantage of such musical structures for an audience unaware of what they are listening to. For my part, I will continue my research into the sonification of FS and GR until I am no longer able, as there is nothing else that brings me greater joy and intellectual stimulation in music. My research in the sonification of FS and GR has never once been boring.

7.6 Possible Improvements to the Test

This experiment was carried out in a highly controlled manner. However, some possible minor improvements to the experiment that could have been made are the following:

- It would have been useful to keep track of the order in which sonifications were presented in the pairs to see if there was an effect on the aesthetic pleasantness when a sonification was played first or second. A small number of subjects mentioned that this could potentially affect the results of the experiment. They nonetheless gave conflicting accounts, with some saying that they always seemed to prefer the first sonification and others saying that they always seemed to prefer the second sonification in a pair. In any case, this could have been a further research question had the program kept track of the data: When presented with a pair of sonifications, does the order in which the sonifications are presented have an effect on their perceived aesthetic value?

- It would have been useful to provide the subjects with an example pair of sonifications for training, so that they would have known more what to expect when presented with the sonifications in the testing portion. This would have lessened the initial shock of hearing the sonifications, which could have had some impact on the evaluations of some of the first pairs presented in each test.

- Rooms with greater acoustic isolation could have been used. Despite the fact that headphones were used in the experiment and the doors remained closed, under certain circumstances, it was possible to hear a sonification being played in the other room, which would have caused some distraction.

8 Conclusion and Future Research

For centuries, mathematicians from diverse cultures have been fascinated by FS and GR. Scientists, philosophers, and artists have taken interest in the mathematical and aesthetic virtues of FS and GR since the mid-nineteenth century, when Zeising made extraordinary claims about the pervasiveness of GR in nature and art. Fechner's skepticism about Zeising's claims about the aesthetic value of GR in visual art led him to develop pioneering research techniques in the field of experimental aesthetics that are still widely used today. Fechner's experiment with rectangles was the first step in determining the aesthetic value of FS and GR in the visual domain. Many researchers followed suit and created their own experiments in the aesthetic value of FS and GR, some of which contradicted Fechner's findings and others which seemed to confirm them. Conspicuously absent in this area of research is an investigation into the aesthetic value of FS and GR in the audio domain or musical composition. Even the earliest researchers covered in this dissertation, Zeising and Fechner, discussed the potential aesthetic value of FS and GR in the audio domain. One likely reason for the lack of research in this area has been a lack of suitable stimuli for research purposes. In the visual domain, researchers have used simple geometric shapes in their experiments in order to isolate factors that could have an effect on the aesthetic value and in order to mitigate the effect of combinatorial and associative influence. Our survey of the utilization of FS and GR in classical composition has shown an abundance of combinatorial and associative influences present in composers' works. If such compositions were to be utilized in research into the aesthetic value of FS and

GR, these influences would encumber a determination of what factors contributed to the aesthetic pleasantness of a piece of music. In order to create the necessary stimuli for research purposes, it is necessary to develop a method of strict sonification that places the mathematical objects to be studied at the forefront of the compositional process, in which all extraneous variables are minimized. Only then can we begin to answer the question of the aesthetic value of FS and GR in the audio domain.

8.1 Contributions

Various contributions to research into the aesthetic value of FS and GR were made in this dissertation. These contributions range from a detailed depiction of how FS and GR have been utilized by composers to empirical evidence suggesting that FS and GR hold no special aesthetic significance or advantage in the audio domain.

8.1.1 The Use of FS and GR in Composition

The following are some of this dissertation's original contributions to our understanding of how FS and GR have been used in composition, and how traditional compositional methods have failed to provide the necessary example stimuli for the purpose of research into the aesthetic value of FS and GR in composition:

- A theoretical framework was developed for the analysis of the use of FS and GR in composition and various new terms were introduced: Fibonacci pitch and pitch class sequences, Fibonacci attack and pulse summation, metric summation, reversal with and without repetition, reversal with change of direction.
- The most comprehensive analysis yet carried out on the utilization of FS and GR in classical composition was presented, providing insight for the first time into what trends have emerged in their utilization: the prevalence of temporal applications in various guises, the rarity of direct applications of GR in comparison to FS and of applications of FS to pitch, the ubiquitousness of the technique of reversal, the apparent absence of the application of FS to dynamics or spatialization.
- Obfuscation of FS and GR and the prevalence of a wide variety of extraneous combinatorial and associative influences were found to be abounding in the works analyzed. Criteria for direct sonification were introduced in order to create the necessary simple example stimuli for research purposes; the ideals that these criteria represent are manifested in a new style of music characterized by mathematical objects fundamentally related to FS.

8.1.2 Experimental Aesthetics and Psychometrics

This dissertation provided initial insight into the main research question of the aesthetic value of FS and GR in the audio domain:

- Results were presented of the first experiment ever carried out in the aesthetic value of FS and GR in the audio domain. Evidence suggested that the utilization of FS and GR could actually have a negative impact on the aesthetic value of sonifications. For two of the five pairs of sonifications presented, evidence suggested that the sonification not closely related to FS was more aesthetically pleasing to the population represented by the sample. For the other three pairs of sonifications, the results were inconclusive.

- The experiment provided insight into the ability of subjects to judge the relative aesthetic value of similar-sounding sonifications with consistency. Evidence suggested that subjects are considerably more consistent in their choice of which sonifications they find more aesthetically pleasing in a pair than would occur purely by chance.

8.2 Limitations of This Research

The question of the aesthetic value of FS and GR in the audio domain is dauntingly general, just as it has proven to be in the visual domain. To address this question more conclusively, a much larger number of sonifications and ascertainments would be required. The fact remains that, as Fechner noted, the same degree of accuracy in results cannot be obtained in the field of aesthetics as compared to physics and various other scientific disciplines. The present study can only provide the first piece of the puzzle, similar to Fechner's first study with rectangles in the visual domain.

8.3 Concluding Words and Future Directions

The research described in this dissertation has demonstrated the utility of sonification as a tool for scientific research in aesthetics. Various questions remain and others might be posed. The following points and questions might provide an impetus for further research:

- We used five pairs of sonifications in this study; one was closely related to Fibonacci, the other was not. If we sonified the same exact mathematical objects in a different manner, would we obtain similar results about the potentially negative impact of the use of Fibonacci-related mathematics in sonification?

- If we chose a number of new pairs of mathematical objects where one was closely related to Fibonacci and the other was not, would we obtain similar results?

- In the general sense, this study utilized the method of choice described by Fechner. What would the results be if one utilized the method of creation in an experiment in the audio domain with the same research questions?

- The subjects in this experiment for the most part had no knowledge of the mathematical objects used in each sonification. What is the effect of knowledge about the mathematical objects sonified on the perceived aesthetic value of the sonification? Vogt, Goudarzi and Parncutt carried out work with a similar

research question [123], nonetheless with a very small number of subjects (20 or less) and with data that was not purely mathematical in its conception (converting images to sound, sonification of a twitter feed and of neurological data). The researchers in this study failed to reject the null hypothesis that there is no effect of knowledge of what is being sonified on the resulting aesthetic value of sonifications.

- In this dissertation, we studied the aesthetic impact of the utilization of FS and GR in sonification in the audio domain. Would we see differences in the results if we provided synchronized visualizations with the sonifications?
- As mentioned earlier, the experimenter failed to keep track of how many times the first sonification in a pair was chosen versus the second. What is the effect – if any – of the order in which sonifications are presented on their perceived aesthetic value?
- Considering that we only obtained statistically significant results for the two pairs that utilized frequency as a dynamic parameter, we could pose the question: What is the effect of frequency on discernability in sonification? Also, would we have obtained similar results if we had used the same mathematical objects for these pairs and not used frequency as a dynamic parameter?

A On the Use of the Word *Style*

The definition of the word *style* is a problem which has long engaged art historians, musicologists and philosophers [124]. The term has been used in various ways [125] and with various ideological underpinnings. Leonard Meyer, for example, defines style in his *Style and Music: Theory, History, and Ideology* analytically [126]:

> Style is a replication of patterning, whether in human behavior or in the artifacts produced by human behavior that results from a series of choices made within some set of constraints.

George Dickinson stresses a more holistic approach in his *A Handbook of Style in Music* [127], stating that musical style can capture aspects of cultural or personal identity [128].

The field of literature seems to be particularly preoccupied with the discussion of the meaning of the word style, and the philosophical issues authors face are equally relevant to artists in other fields. Their theories have been divided into three categories: rhetorical dualism, psychological monism and Crocean aesthetic monism. According to Louis Milic [129], in the theory of rhetorical dualism, the oldest school of thought regarding style, ideas can exist independently of their expression. The intention of the writer is prior to his expression of it, and thus a single intention can be expressed in a multitude of ways. The theory of psychological monism assumes iterations of style are the expressions of personality and cannot be chosen any more than one's handwriting or temperament. This corresponds to Comte de Buffon's (1707–1788) celebrated quote "the style is the man himself." The essence of the third theory, Crocean aesthetic monism, lies in that it denies outright any separability between form and content; any discussion of style is useless and irrelevant, as the work of art is a unified whole with no separation between meaning and style. Both monistic theories seem to suggest that style cannot exist a priori, and that it is only recognized a posteriori as a by-product of an artist's work.

Considering these widely varying viewpoints, in my opinion, the historical disagreement and pluralism in the definition of style is a reflection of the inadequacies of language and our attempts to overcome this problem. I view these theories as more complementary than contradictory; rather than getting bogged down in any particular dogma, the emphasis should be on achieving interesting and aesthetically pleasing results. It is my conviction that it will never be possible to formulate a definition of style which would be agreed upon by all artists. This is true for the definition of the words *music* and *art* as well, which have also been wrought historically with deep philosophical disagreements. Perhaps more interesting, therefore, than attempting to find the "true" definition of style, which is clearly unlikely to be agreed upon, are the effects one's own conception of style can have on one's artistic work. What can one gain artistically, for example, in believing that a style can or cannot be conceived a priori? My own philosophy has been dualist in the sense that, for me, a style can exist conceptually in an artist's mind prior to realizing works in that style. In this sense, style can also exist in the conceived set of techniques, material, manner of expression and – most importantly – imagined results. For better or for worse, both

psychological and Crocean monism seem to discount an artist's power of imagination and foresight, rather focusing sole attention on the end result. I agree with Milic that especially in Crocean monism, the very concept of style seems to dissolve completely in favor of the concept of the end result – or else it would seem that every work would have to be in a different style, which seems to render the word *style* practically useless as a means of categorizing works of art.

B Translated Excerpts of Zeising's and Fechner's Work

This section contains various translations of excerpts from Zeising's and Fechner's most important and relevant work on the subject of aesthetics as well as FS and GR [6, 7, 8, 9, 10, 11]. They are ordered chronologically and according to page number. All translations are by the author. Indentation follows the originals and *italics* were used to substitute e x p a n d e d t e x t in the originals.

B.1 Adolf Zeising

B.1.1 *Neue Lehre von den Proportionen des menschlichen Körpers* (1854)

New doctrine on the proportions of the human body, developed based on a fundamental morphological law pervading all of nature and art that previously remained unknown, and accompanied by a complete historical summary of previous systems by Professor Adolph Zeising. (A. Zeising, *Neue Lehre*: title page)

As such, I present this book to the public with the equally humble as well as joyous awareness in order to have given solely the initial impetus for further research and more comprehensive investigations. And I feel obliged in regards to the little that I myself have been able to contribute to the topic to request that one indeed judges the *subject* with full scientific stringency, nonetheless, the *approach* to such with the same hindsight that every first exposition of a new idea may make claim to for itself. (A. Zeising, *Neue Lehre*: XII)

The concept of proportionality is connected most intimately with the concept of beauty; to clearly recognize one separated from the other is impossible. This is not at all to say, however, that both of these concepts are identical, nor that proportionality holds the same degree of importance in all beautiful manifestations. Beauty in general is an *aggregate* of qualities consolidated into unity by the observing mind; proportionality, however, is only an *individual* one of these among others. The highest beauty is that in which all of the contributing qualities for the engenderment of beauty are unified in the most perfect harmony. This is only possessed, however, by the *absolutely* beautiful manifestation: i.e. the world. (A. Zeising, *Neue Lehre*: 131)

And indeed the merits of this proportion above all others are immediately apparent to the eye, even when one considers them purely from a mathematical perspective. Not only does it possess the merits of all continuous proportions, but supercedes every *other* continuous proportion 1) because it does not merely create a mediation between two sizes randomly brought together, but between the whole and its smaller part, such that for this reason its underlying ratio is not a random or changing – and in and for itself perhaps highly disproportional – ratio, but rather a necessary one, constantly remaining the same everywhere and moderating, regardless of how small or large the whole to be divided may be; 2) because the two smaller parts taken

together are always equal to the greatest part, i.e. the whole, and because the smaller part therefore is the complement of the larger, just as the larger is conversely the complement of the smaller. The proportion is therefore not only a perfect *geometric* proportion, but in a certain sense an *arithmetic* one as well, because its parts represent not only factors of the same products, but also both of the complementary summands of a sum. (A. Zeising, *Neue Lehre*: 163)

As we shall now proceed to solve this problem, we will derive the evidence first from the sphere of manifestations that are *visible* or present themselves *spatially*, and then from those that are *audible* or present themselves *temporally*: For it will show itself to be true that the law that governs the *proportionality of the body* is indeed that which also governs the *harmony of the tones* – so that not only the domain of *sculptural* perception, but also that of *sonic* perception is governed by it, and hence, among the arts, not only architecture, sculpture and painting should take interest in it, but rather music and poetry as well. (A. Zeising, *Neue Lehre*: 174)

Up until this point we have considered the law purely in its generality. Of course, it does not occur as such in the individual, real phenomena anywhere, rather every such phenomenon deviates to some greater or lesser degree – indeed each in its own unique individual way. (A. Zeising, *Neue Lehre*: 296)

The question now presents itself: which relationship among tones is it that corresponds most perfectly to this ratio? In order to answer this, we need only glance over the above listed intervals again and compare them with the sequence of our ratios (p. 167) in order to immediately recognize that it is those two relationships among tones that express the ratios of 3:5 and 5 to 8: for the rounded numbers 3, 5, 8, 13 are only slightly different from the following four members of our sequence:

$$3.1056272,\ 5.0249943,\ 8.1306215,\ 13.1556158$$

and the two proportions

$$3{:}5 \approx 5{:}8 \text{ and } 5{:}8 \approx 8{:}13$$

can thus be created from these, which come so close to the requirements of our law that, in the first of the two, the product of the two extreme members (24) is only *one* twenty-fifth smaller than the product of the two middle members (25); in the second, however, the product of both of the middle members (64) is only superceded by the product of both extreme members (65) by only 1/65.

Through the more exact division of the numbers 8 and 13 we get the following two proportions:

$$3.06{:}4.94 \approx 4.94{:}8 \text{ and } 4.98{:}8.02 \approx 8.02{:}13$$

As such, the rounded number 5 is, when it is taken as the median proportional number of 8, only 6/100 too large when in comparison to the more exact expression 4.94; in contrast, the number 8, when considered as the median proportional number to 13, only 2/100 too small in comparison to 8.02. Both numbers only deviate in such a way by a very small amount from the ideal middle, in which even less offense is to

be taken, as real phenomena in general never reach the idea, and certain deviations from the idea are even necessary if the inner richness of the idea is to come to light in multifarious manifestations. Since (if we retain the rounded numbers for the sake of simplicity) the ratio of 3:5 corresponds to the major third with the octave of the lowest note, e.g. in the C major scale the relationship between e and c′, and in contrast the ratio of 5:8 corresponds to the *minor* third with the octave of the lowest note, e.g. within the C major scale the relationship between d♯ (e♭) and c′, as such the minor and major sixths must be seen as the consonances corresponding to our proportional law as calculated from the top downward, i.e. e+c′ and e-flat+c′. For the following proportions hold:

$$\text{e:c}' \approx \text{c}':\text{e+c}' \text{ and } \text{e}\flat:\text{c}' \approx \text{c}':\text{e}\flat+\text{c}'$$
$$5{:}8 \approx 8{:}13 \quad 3{:}5 \approx 5{:}8$$

The fact that these two relationships among tones are really the most pleasant and satisfying among two-voiced relationships follows irrefutably from the fact that they are the only intervals that can conclude a musical period, for which reason improvised two-part folk song and the simple music of two horns as well proceed in sixths and their complements, the thirds. (A. Zeising, *Neue Lehre*: 430–431)

The relationship of the octave to the *major* of the lowest note thus corresponds to the ratio of the upper body to the lower body in the *male* form, and the relationship of the octave to the *minor* third of the lowest note corresponds in contrast to the ratio of the upper body to the lower body in the *female*. The former chord thus has the character of greater *austerity* and *harshness* and is therefore called *durus*; the latter chord has the character of greater *favorableness* and *softness* and is called the *mollis*.[17] (A. Zeising, *Neue Lehre*: 437–438)

B.1.2 *Aesthetische Forschungen* (1855)

Wherever something beautiful comes to being, it always presents itself as the product of two factors opposing each other that are, however, in an interdependent relationship. Namely, on the one hand, of an observing, perceiving *subject*, which as such belongs to the *ideal* world, and on the other hand of an observed, perceived *object*, which as such is rooted in the *real* world. Indeed, strictly speaking, beauty is actually nothing but the vivid back-and-forth flux between the subject and object, between the ideal and real, in which the difference between the two is overcome in such a way that they are perceived as one in the midst of their opposing relationship and in this unity as an aggregate of everything ideal and real – in short, as the absolute, immaculate state of being.

With this unique essence of beauty, the problem that the science of the same has to solve cannot consist of anything other than, on the one hand, a definition and delineation of beauty, which is as clear as possible, according to its *concept* – namely from the perspective of its *ideality* – and on the other hand, an observation and

[17] In German, the "Durakkord" and "Mollakkord" refer to major and minor chords, respectively. In this translation, the Latin adjectives from which these terms are derived have been used.

analysis of such, which is as exact as possible, according to its external *appearance* – namely from the perspective of its *reality*. And for this purpose, thirdly, the proof must be furnished that there is indeed such a correspondence and analogy between the real and ideal qualities of beauty, if not a complete identity, according to which the one is to be regarded as images and reflexes of the other, and both are indeed to be considered in a certain regard to be emanations of one and the same focus diverging, nonetheless directly together in their origin, and converging again at their destination.

As obvious as it seems to recognize this as the task of aesthetics, those who have worked in this area previously indeed did not become fully aware of this, or they at least lacked an approach to the subject corresponding to this realization. Indeed, for a long time one has correctly recognized that beauty is based on a correspondence of the real and ideal. However, instead of demonstrating on the one hand the real and on the other hand the ideal according to exact research what the correspondence of one to the other consists of and by which qualities the individual, finite manifestation could prove to be an image of the all-encompassing, infinite state of being, one considered this correspondence to be a clandestine fact not to be further explained and considered it no longer necessary to concern oneself, in addition to the purely idealistic conception and development of beauty, with the exploration of its real qualities existing in space and time as well. (A. Zeising, *Aesthetische Forschungen*: III–IV)

In this respect, this writing represents as such a first attempt to consider beauty more than has yet happened also according to the manner of conception of the natural sciences and to trace the causes rooted in the spatial and temporal ratios, in material and formal conditions from which the various aesthetic effects of real manifestations emerge. (A. Zeising, *Aesthetische Forschungen*: V)

Nonetheless, I dispatch the book with trust to the public and believe even more to be permitted to expect a friendly and indulgent reception on the part of the audience and criticism, since one will at least not fail to recognize in it an honest aspiration, a warm devotion to the subject and a new, independent approach to the topic, and since I have already enjoyed the pleasure that my writing "Neue Lehre von den Proportionen des menschlichen Körpers" (Leipzig, 1854), rooted in the same principles as this writing, and the discovery established in it of an aesthetic-morphological proportional law pervading all of nature and art (see §168 in the following), has become part of a lively discussion in natural scientific as well in artistic and philosophical circles. (A. Zeising, *Aesthetische Forschungen*: VII–VIII)

The most important and fundamental reasons for which I cannot yet consider the aesthetics of the Hegelian school to be satisfactory are the following:

1. The dialectical derivation of beauty as well as the entire developmental process of Hegelian philosophy are based on untenable categories.

2. The relationship of beauty to God, to the world, to nature, to the mind, to history etc., quite particularly, however, to the true and good is defined in part inadequately and inexactly, in part virtually incorrectly.

3. In the definition and development of beauty itself, neither the scope nor the content of beauty is done sufficient justice to, and in particular, a more exact exploration of the *objective elements* and *qualities* of beauty, a satisfactory *classification* of beauty into its various *modifications*, and finally a sufficient presentation of the various *manifestations* or *forms of embodiment*, which beauty takes on as a result of the realization within the various spheres of manifestation, are lacking. (A. Zeising, *Aesthetische Forschungen*: 28)

When in the preceding text the ratio of the golden section is declared to be the most perfect and satisfying, this is of course not to claim as such that it is absolutely the one and *only* of aesthetic significance, rather only that it is the one that proves to be that which those remaining strive towards according to their ideal and that on which formal development, when it has more or less reached its ideal, most often tends to be based. (A. Zeising, *Aesthetische Forschungen*: 198)

If one seeks the corresponding pitches for the numbers of the sequence converging from unity (1:1) to the ratio of the golden section (3:5, 5:8 etc.), then one obtains the following:

$$\text{C:c:g:e}'\text{:c}'' = 1{:}2{:}3{:}5{:}8$$

in which one will immediately recognize the pitches of the four-part chord – namely in the combination that is most appealing to the ear – and see at the same time that it consists of the intervals of the octave, the fifth, the major and minor sixth, while it contains in its normal ordering (C:E:G:c = 4:5:6:8) the intervals of the major third (4:5), the minor third (5:6) and the fourth (6:8) – namely the transpositions or complementary intervals of both sixths and the fifth. (A. Zeising, *Aesthetische Forschungen*: 247)

B.1.3 *Das Normalverhältniss der chemischen und morphologischen Proportionen* (1856)

With its small scope, this writing can of course only have the purpose on the one hand of resuming in a short summary that which is already to be considered certain about the significance of the mathematical ratio covered in it, and on the other hand to call attention to new areas in which the ratio seems to be that which determines measure and form. (A. Zeising, *Das Normalverhältniss der chemischen und morphologischen Proportionen*: V)

The ratio of the golden section – which will be discussed in this writing, and which I extensively accounted for and covered in my "Neue Lehre" and would like to summarize here, is that ratio that, as a standard ratio, underlies the arrangement of the human form, the construction of more noble animals, the construction of plants, namely in regards to their leaf-arrangement, the forms of various crystals, the arrangement of the planetary system, the proportions of architectural and sculptural works of art recognized as being the most beautiful, the most satisfying chords in musical harmony, as well as many other things in nature and art – consists of the

division of a measurement given as a whole into two unequal parts such that *the smaller part (lesser) is to the larger part (greater) as the larger part is to the whole, i.e. the sum of both parts.* (A. Zeising, *Das Normalverhältniss der chemischen und morphologischen Proportionen*: 1–2)

B.2 Gustav Theodor Fechner

B.2.1 *Ueber die Frage des goldnenen Schnitts* (1865)

How does it look with the application of the now so widely discussed golden ratio to painting? Zeising, the discoverer of the aesthetic significance of the golden ratio, considers the Sistine Madonna to be the most outstanding example of such. (G. T. Fechner, "Ueber die Frage des goldnenen Schnitts": 100)

However, all of that again does not fit at all; and I searched for the golden ratio in such a way completely in vain until I finally discovered that it is indeed not found *exactly*, as Zeising says – for in matters regarding the golden ratio one mustn't take Zeising's "exactly" too exactly – but indeed very closely approximated when one sets the lower border at the lower end, i.e. the right tip of the foot, of the Madonna figure, where clearly no one can set it who is searching for the golden ratio for the *entire* group. (G. T. Fechner, "Ueber die Frage des goldnenen Schnitts": 102)

I doubt that one can accordingly speak of a confirmation of the rule of the golden ratio in the Sistine Madonna, since it is only found with an inappropriate manner of application of the measure, and even then only with an approximation that one cannot consider to be sufficient. (G. T. Fechner, "Ueber die Frage des goldnenen Schnitts": 102–103)

Accordingly, one might say that Raphael sought much more to avoid the golden ratio rather than to abide by it in the most visible vertical division of his groups for observation … (G. T. Fechner, "Ueber die Frage des goldnenen Schnitts": 105)

Zeising, however, indeed finds the most extensive confirmations of [the law of the golden ratio] in the measures of the human body, the most beautiful statues, the most beautiful buildings[18] – indeed everywhere in the heavens and on earth. If only the manner in which he finds them did not have too much similarity to the manner in which he finds them in the Sistine Madonna. According to a similar method of research others have found simple rational ratios, which produce consonances in music, instead of the irrational golden ratio as decisive for the beauty of the human body and buildings, and in such have taken into account philosophy and experience as well as Zeising, and I do not find that his method of research has a principle advantage over theirs. More on this elsewhere. (G. T. Fechner, "Ueber die Frage des goldnenen Schnitts": 105)

[18]The German word "Bauwerke" has been translated as *building* here and elsewhere, although this refers more generally to anything that could be a project for construction.

I consider such complicated examples as those that Zeising claims to be proof of the aesthetic and other significance of the golden ratio to not be suitable as proof in the first place, and their quantity much more suitable to multiply the stringency error than to eliminate it. (G. T. Fechner, "Ueber die Frage des goldnenen Schnitts": 105)

Clearly, for the same reason that I consider Zeising's evidence as not proving the golden ratio, the counter-evidence mentioned earlier cannot claim to be binding proof of the contrary against it. He could nonetheless be correct. I also only wanted to show that one can find the aesthetic significance of the golden ratio refuted just as easily as proved following the path beaten by Zeising, when one does not merely select or prefer the applicable examples and measures, and assuredness is not to be had at all on this path. For this purpose, I believe, one must operate instead with the *simplest* examples, where the measure can only be applied in a *single* way, and neither the purpose has a considerable influence on the form nor the association of the fulfillment of the purpose or significance on the pleasantness. (G. T. Fechner, "Ueber die Frage des goldnenen Schnitts": 106)

After all of this I have the greatest doubts about the aesthetic significance of the golden ratio as a *ratio of division*; for that which speaks against it above seems to me to be more fundamental than that which Zeising was able to claim. It is different with the golden ratio as a *ratio of dimension* or *aspect ratio* of simple rectangles, and perhaps beyond this. With as inauspicious of prejudice against it that I went into the research, I have indeed convinced myself – through equally simple experiments as those that spoke against the aesthetic significance as a ratio of division – of its significance as an aspect ratio, which is indeed much less than that of symmetry, but indeed is not absent. For symmetry, every single experiment is decisive; for the golden ratio, only the average and the majority of preferences on the whole. This showed itself to be true in part through the presentation of a larger series of rectangles made of white card stock with varying aspect ratios to a few hundred people that the one designed according to the golden ratio was most often preferred as the most pleasant (most often rejected was the square and the rectangles closely approximating it), and in part through gaging the *simplest* applications not considerably influenced by purpose which occur in commerce, consulting all examples that one could obtain and the selection of median results. In contrast, the simple rational proportions that are namely often declared to be the beautiful standard proportions in architecture and in the construction of the human body, which correspond to the musical consonances, have no advantage at all according to both methods of testing. The pleasantness declines according to the measure of distance of the aspect ratio from the golden ratio continually without increasing in any way when passing through simple ratios such as 1:2, 2:3, 3:4. This is according to experiments in which the rectangles were presented for judgment mixed in all possible changing positions. I will not discuss the difference according to whether the rectangles were upright or lying flat here. (G. T. Fechner, "Ueber die Frage des goldnenen Schnitts": 110–111)

As much as I believe accordingly that Zeising's discovery of the significance of the golden ratio must be qualified, I indeed also consider it with such a qualification to be a very interesting discovery – indeed the first that one can truly consider to be a discovery in aesthetics. It will preserve his name in the history of aesthetics. And certainly it was more ingenious to make it than to qualify it. (G. T. Fechner, "Ueber die Frage des goldnenen Schnitts": 111)

B.2.2 *Zur experimentalen Aesthetik* (1871)

This first part of the research offered here will only contain the explanation of the research methods in addition to general perspectives, historical and critical comments. Meanwhile, the necessary rules of calculation for the evaluation of the experiments with the experiments themselves are reserved for the part that follows. (G. T. Fechner, *Zur experimentalen Aesthetik*: 1)

After all my research, when I hold in prospect the beginnings of an exact approach to aesthetics in the following discussion, I do not at all believe that the exactness of the results in this science can be taken as far as they are in astronomy and physics. I do, however, believe that through the application and further development of the methods dealt with here that one can come as close as possible to exactness and assuredness. There are differences in general between various sciences and between various tasks within each individual science in this regard. Generally speaking, physiology can also not compete with astronomy, physics, and chemistry, in regards to the exactness of its results, because it cannot find the conditions to be researched isolated to the same extent or cannot isolated them as easily. And this difficulty of isolation applies to our science as well. (G. T. Fechner, *Zur experimentalen Aesthetik*: 2–3)

The problem that should be taken up next is whether ratios of dimension and divisions of objects have an advantage in pleasantness greater than others and, in the affirmative case, which ones they are. The forms of most objects are contingent upon their idea, purpose, significance – or at least partially contingent. Every form can appeal through its appropriateness for such, and every form can displease through its inappropriateness for such. Namely, the former case when the purpose, the significance of the object appeals to us itself in that the pleasure inherent in it is non-arbitrarily transferred to the form of the object. Since the purposes and significances of objects are of the most various sorts, in such a way the most various forms can be pleasant or unpleasant according to whether they are fitting or not, and accordingly our task appears to be completely undefined. The question poses itself, however, of whether the appropriateness or inappropriateness of a form for something that goes beyond the form itself is the *single* reason for the pleasure or displeasure, not in the contingency of pleasure or displeasure in the formal ratios in themselves, which also require consideration in utilizations and in the aesthetic judgment. (G. T. Fechner, *Zur experimentalen Aesthetik*: 4)

Insofar as the impression that forms, colors, sounds make as well as the relationships of forms, colors, sounds make through their own constitution is in any case to be differentiated from the impression that they make as a carrier of a certain significance in the service of certain purposes or ideas – I call the former the direct impression for brevity's sake, the second the associative impression. Or, insofar as both amalgamate into a unified impression: the former the direct, the latter the associative factor of the impression. And I speak of an associative influence when the latter comes into play with the former. (G. T. Fechner, *Zur experimentalen Aesthetik*: 6)

Our task here is only to investigate what effect the simplest formal proportions *irrespective of associative influence* have in the aesthetic respect, for which it is necessary to find a means to emancipate them from this influence or to compensate for such. The associative influence is then to be investigated in itself in the greatest possible abstraction from the direct influence, and finally the investigation is to be turned to the relationship of their convergence, where both can accordingly work harmoniously or in conflict. Here, however, we are only concerned with the first of these three tasks. (G. T. Fechner, *Zur experimentalen Aesthetik*: 6–7)

Whatever type and from whatever origin the aesthetic impression of a form may be, it can sustain a modification through constitution or combination with other equal or different forms, which I call the combinatorial influence. This can be direct or associative: the former when new conditions of pleasantness or unpleasantness in themselves arise through constitution or combination, as for example when a circle looks better without detectable associative influence in a square than around it; the latter when new significances, relationships to purpose, ideational relationships emerge as a result. Accordingly, something that is pleasant in itself or in a certain combination can seem unpleasant in another, and vice versa, although there are also aesthetically indifferent combinations. If, for example, for a window, when it is observed in itself, a certain ratio of the height to the width has appeared to be the most advantageous, because it coincides in general with the most aesthetically advantageous ratio of an isolated rectangle, it does not at all follow that it lays claim to the same advantage in the combination with the other windows situated parallel to it, and it can be imagined that the ratios of the windows have to suit the facade that they are situated on. The facade, however, is aesthetically influenced by the proportions of the roof. In which case it begs the question in general what is to be gained by recognizing the aesthetic value of a ratio in itself when this value changes in every other constitution or combination. To this, however, one can reply: first, that in many cases of combined forms, a certain ratio indeed determines the main form of the object, and accordingly gains a dominating influence; second, that for many objects, particularly artistic objects, an isolation from outside influences is artificially created through the framing or through intentional creation of a neutral environment, by which means the influence of the combination of its form with others is noticeably removed; third, that for many objects, the environment changes according to chance, which compensates for the influence on a whole. Insofar, however, the influence in question in general indeed is not to be denied, nor underestimated, and is not to be

removed in every case, and is rather to be used in a manner that is as advantageous as possible, the task of our research is not annulled by such, but expanded, in that the influence of the constitution and combination is to be investigated as well. Even if there are infinitely many, there are indeed not infinitely many laws to which this influence is subordinate, and the intent is to be directed mainly towards their investigation. In such, however, the starting point always remains the investigation of the aesthetic value of isolated forms that are as simple as possible, which as such remains, so to speak, central for all of the deviations accounted for through the various constitutions and combinations, and as in accordance emerges more purely, as the formal ratio concerned dominates more and is more perfectly isolated, the combinatorial influence in general recedes more. In our area of study just as in every other, clarity and success is only to be had by investigating what every condition contributes in itself, and what is associated with the combination of such. If it is such in physics, it will also be as such in aesthetics. Clearly, as far as one might take the rules of combination, one will not create any beautiful buildings or beautiful visual art at all according to these rules; only the feeling of the artist is capable of combining *all* of the conditions of pleasantness inherent in the purpose of the task into the most advantageous result. But is this, however, perhaps different in musical composition? And is for this reason a study of scales, dissonant and consonant chords, the resolution of dissonances, the meter and rhythm etc. superfluous? It would be tedious to look into the reason why it is not superfluous. It is sufficient that one can make use of it in our area.

In such, it comes into consideration that the associative factor in music in general only plays a supporting role, the direct factor the main role. Meanwhile, in the higher visual arts, the main role is attributed to the associative factor. But the difference is only relative. For in painting and sculpture as well, not *everything* can be attributed to the associative factor, and the tectonics, in which, generally speaking, both factors hold the balance or change with dominance, such as in ornamentation, where the direct, generally speaking, has dominance, both factors need to be taken into account down to the most elementary conditions as well. (G. T. Fechner, *Zur experimentalen Aesthetik*: 7–9)

According to the such extraordinary significance which the golden ratio has, one must clearly be astonished that it was able to remain hidden for so long, and this might be suited to raise some concerns right from the start. Indeed, even if Zeising finds implications of the golden ratio in multiple constellations, in the distances between planets and the arrangement of land and sea on earth etc., even traces of it "in purely spiritual spheres of science, as well as in ethical and and religious regards" and takes into account in addition everything which only seems to mildly fit, everyone will say to themselves right from the start that his doctrine *at least* must be damaged by exaggeration and a lack of criticism. (G. T. Fechner, *Zur experimentalen Aesthetik*: 14)

Zeising's aspiration to establish the recognition of the golden ratio has clearly had little success with natural scientists. With few exceptions, one has not cared for his

general formal principle or nature. Nonetheless, his doctrine is taken into account substantively in the handbook on anatomy of Harless, and Dr. Hagen claims to have found the axiom of the golden ratio confirmed in the greatest detail by his measurements in cephalic and brain construction. Also, the assimilation of some of Zeising's essays into magazines on natural science at least proves that one has not rejected their place everywhere. (G. T. Fechner, *Zur experimentalen Aesthetik*: 15)

Because he is of the opinion, however, that the same principle of pleasantness must be valid in both the field of vision and sound – and does not want to hear of anything at all that deprives the golden ratio of power – he finds himself obligated as a result, clearly in contradiction to his general judgment, to declare both sixths to be the most musically pleasant intervals, for which he believes to find reasons in musical compositional ratios (see p. 414 ff. in *Neue Lehre*). For pure consistency, he would clearly be forced instead of the pure sixth, to declare the impure sixth which corresponds *exactly* to the golden ratio, to be the very most pleasant musical ratio. And since this, nonetheless, does not work well, Zeising takes refuge here, just as everywhere else where the golden ratio only fits with a coarse approximation, (*Neue Lehre* p. 431) in the sentence, which is clearly to be acknowledged to a certain extent: "as real phenomena in general never reach the idea, and certain deviations from the idea are even necessary if the inner richness of the idea is to come to light in multifarious manifestations." One can only ask how a ratio is supposed to be an ideal ratio for pleasantness if deviations from it supercede it in pleasantness. (G. T. Fechner, *Zur experimentalen Aesthetik*: 16–17)

My judgment in this regard is very unbiased. From the start I admit going into the research with the greatest mistrust for an aesthetic significance of the golden ratio *in general*: first, because it seemed strange as already mentioned that its significance would have proved valid long before Zeising if it were so great as Zeising claims after symmetry, much less valued by Zeising, has always been perceived; second, for the mentioned critical considerations. And so I believed to be able to easily disprove Zeising's belief by means of the simplest experiments according to the methods to be discussed. I found myself surprised, however, to find them much more aptly confirmed in regards to ratios of dimension. Only the advantage of symmetry remains much more decisive than that of the golden ratio such that small deviations from symmetry not stipulated by purpose or significance are perceived to be much more unpleasant that those from the golden ratio, which one would not expect according to Zeising's presumption about the golden ratio, which meanwhile explains in itself why its value remained unknown for so long. (G. T. Fechner, *Zur experimentalen Aesthetik*: 23–24)

As a ratio of dimension, only *one* ratio has a fundamental aesthetic value in itself that emerges simply in the dimensions of simple rectangles: the golden ratio, in that according to measure as it is deviated from, the ratio becomes more unpleasant without any accompanying increase at all in the pleasantness curve with simple rational ratios occurring in the changing 1:1, 1:2 etc. Indeed, the simplest ratio on the one end of variation of 1:1, which produces the square as a ratio of dimension, apportions itself

along with the next closest, very short ratios and very long ratios tending towards the other end (1:∞) with the greatest disadvantage. Similarly, only *one* ratio has a fundamental aesthetic value as a divisional ratio: the same which is among the most disadvantageous as a ratio of dimension – the ratio 1:1, which produces symmetry and emerges purely and clearly free from associative influences on a simple horizontal line in that, according to measure, as it is deviated from, the ratio becomes more unpleasant without an increase in pleasantness accompanying in this change with the golden ratio occurring. However, through associative and combinatorial influences, the previous axioms can sustain alteration whose general laws remain to be studied. However, the ratio of 1:2 seems to show a significant advantage over the golden ratio according to my experiments carried out up until this point, although not yet sufficiently expansive in this regard, under conditions which favor or call for an inequality in division – which, if it allows for a more general and more certain expression than I am currently able to provide, could be viewed in a certain manner as an expansion of the principle of simple rational numbers, and would imply that the simplicity with which we divide with our eye the division of 2 into two subdivisions, each division of which corresponds to 1 – through which symmetry is created in our imagination – plays a role in this advantage. (G. T. Fechner, *Zur experimentalen Aesthetik*: 29–30)

Given it was supposed to be proved that symmetry in itself is more pleasant than asymmetry – in such one should not look to buildings and the human body; there the proof might always remain doubtful, as much as one wanted to measure. In the horizontal direction, symmetry is inherent in these examples, in the vertical it is absent – in exceptional cases even horizontally. And it remains questionable what to attribute to pleasantness in itself and what to utility and other influences. However, one can make the determination through the simplest presentation in comparison of symmetrical and asymmetrical figures from which everything is isolated as much as possible besides symmetry and asymmetry that can have an influence on the preference of pleasantness. Every child who is not all too young will prefer simple symmetrical figures to asymmetrical, and prefer the kaleidoscopic arrangement of colorful stones to their random arrangement – every adult will agree. Why does one not make the same determination about the preference of the golden ratio or simple rational ratios over other ratios? If the golden ratio is equal to symmetry in pleasantness or even supercedes it, then this must indeed also reveal itself in experiments with simple rectangles and division of simple lengths or strips where no choice between a certain method of layout and no influence of consideration of purpose etc. occurs, or all philosophical explanation and emperical appealing to complicated examples is of no avail.

Even if the preference of direct pleasantness of evenness or unevenness of rhythm, i.e. the rule of regularity over irregularity in the repetition of sonic impressions or movements was to be proved, appealing to the benefit of regularity in music and in dance would not be decisive since there are, in contrast, some activities that according to their nature are performed better without a defined regularity – where the sought execution of regularity is much more unpleasant, even in free spoken discourse; accordingly, one could contrarily conclude the unpleasantness of regularity.

However, the simplest comparison of a meaningless pulse – whether it be rhythmically structured or even completely uniform – with a temporally irregular changing beat or noise (for which purpose regular and irregular drumming with the fingers can serve), leads to a determination where it is no longer a question of whether it is the going into an activity – arousing the aesthetic feeling from other perspectives – and the promotion of this activity or the activity by the regularity itself that makes it emerge as more pleasant than irregularity.*) If the determination is made, however, for the regular pulse free of aesthetic influences in such a way, one can then predict its performance when such aesthetic influences come into play – not simply by just taking into account what it accomplishes in itself and what the influences accomplish, but through the employment of the aesthetic helping principle, which will accordingly increase the pleasantness of every activity in which it has a promoting or agreeable effect even more than one could predict according to its abstract performance. Meanwhile, its pleasantness will otherwise easily be superceded by the unpleasantness of the impedance of the activity.

*) One might notice to the contrary that a continuing empty regular pulse is exhaustingly boring. This is true, but the most beautiful work of art bores one as well should one continuously observe it. An irregular, meaningless beat changing against the pulse, however, not only becomes boring in time, but unbearable and gets on one's nerves. As such, even if one expands the comparison of regularity with the irregularity of noises or beats for a short or long term, the former will always have the advantage and this advantage will be verifiable without one comparing a regular *activity* with an irregular one. (G. T. Fechner, *Zur experimentalen Aesthetik*: 41–42)

One cannot directly measure the degree of pleasantness or unpleasantness to which a ratio appears to the former or latter under the former or latter circumstances; such would require a unit of measurement of individual like or dislike, which we do not yet have. However, one can convey the measurement in a different way in that one *counts* the persons who prefer the one ratio or those who the others in the simultaneous presentation of such or, in general, equal possibility of selection, and that ratio that is primarily most pleasant to the class of people that one consulted in the experiment explains which gathers the most votes of preference; or, if this does not exist, such a ratio around which the votes of preference most densely cluster determines, in general, however, the degree of the relative pleasantness of a given ratio for a given class of persons according to the relative number of the votes gathered for it in this class. To what degree such a determination can claim a more general authoritative significance will be discussed further on.

No objection to this measurement of pleasantness from aprioristic perspectives can be raised, as it is already determined a priori by definition itself. I not only namely *call* something according to measure because it gathers more votes of preference for itself under comparable circumstances, but also expressly attribute to it the *concept* of greater or lesser pleasantness in addition. And I justify this on the one hand because it truly corresponds to the natural use of language and terminology to declare something as more pleasant that has the characteristic of pleasing more, and on the other hand because of the possibility offered, realizable up until this point in

no other way, to introduce the concept of measurement into aesthetics in general, and finally because of the advantages of the introduction of this measurement. For in any case, a fundamental piece of data of fundamental importance for aesthetics is to know in what relation the votes of preference for every value of measure of a ratio to be evaluated for its pleasantness are distributed. And as such it serves only brevity – instead of saying with many words, for example, that a ratio gathers twice as many votes of preference as another – to say that it is found to be twice as pleasant. Also, the measures obtained in such a manner cannot be introduced any less into calculation than any other measures, and be utilized through such means; meanwhile, other methods of expressing greater or lesser pleasantness remain completely indefinite. (G. T. Fechner, *Zur experimentalen Aesthetik*: 44–45)

The tasks that we can present ourselves with here are connected to the previous definition of our concept of measurement. *First*, it will have to be researched among the values of a formal ratio with underlying variation independently from concrete aesthetic influences whether there is a majority of votes of preferences (not simply dependent on unbalanced randomness) at all, and as such a majority of preference above the values deviating from it, and which one it is; *second*, to define the decline in pleasantness according to the decline in the number of preferences for a given deviation from the most pleasant value and, if possible, to find the law for the decline; *third*, to derive from the measures a few other main values for the judgment of aesthetically pleasant ratios besides the most pleasant value, which lays claim to the most interest, which will be covered later, together with the relationships between them; *fourth*, to determine the degree of accuracy of the results. In addition to these tasks, which I succinctly summarize as *objective* tasks, one can also assign oneself a few tasks which I call *subjective*, to be discussed below.

Three methods are available to us to accomplish these tasks, which I succinctly differentiate as *method of choice*, *method of creation* and *method of utilization* and characterize next according to their main aspects as follows.

Method of Choice. One presents the ratios to be compared in terms of their relative pleasantness in as simple of schemes as possible to many people and has them – clearly admonishing them to not think about any specific application, when it is about direct pleasantness – designate the ratio that emerges as most pleasant to them according to its own character and the one that is most unpleasant, and records every judgment of preference and rejection in a column of a special table of preferences and rejections with the pertaining ratio; namely with 1 when the preference or the rejection is for *one* specific ratio, and in contrast with $\frac{1}{2}$ or $\frac{1}{3}$ if there is fluctuation between 2 or 3 ratios, in order to eventually count up and compare the numbers in every column of the table of preference and rejection, in which the less important (also not often utilized by me) table of rejections indeed affords a sort of control, as the most rarely preferred ratios must be those most often rejected if the method is to be sound in itself.

Method of Creation. One arranges for many people, instead of *choosing* the most pleasant among multiple presented ratios, rather to *create* them according to the simplest possible scheme themselves, in which one investigates for which ratios the

most subjects coincide or around which the individuals cluster most densely; the lesser degrees of pleasantness is measured, however, according to the lesser number of those that remain with a given ratio.

Method of Utilization. One measures the dimensions or divisions of the *simplest* objects occurring in use, commerce, doings and dealings – in short, I mean in life – for which the form is determined much more directly through consideration for pleasantness (assuming one is dealing with pure, direct pleasantness) than for purpose, the significance or the appropriateness for other forms, and which allow for no arbitrariness in the manner of application of the measure, where one again determines the relative measure of pleasantness by means of the relative frequency of occurrence of one formal ratio or another. (G. T. Fechner, *Zur experimentalen Aesthetik*: 47–49)

It would probably often be useful, however, to lend a more secure foundation than the more or less subjective individual connoisseurs' votes to the commonly occurring argument about which of two works of art is more beautiful by means of the statistical statement of the first method, which one receives more votes of preference in authoritative circles. (G. T. Fechner, *Zur experimentalen Aesthetik*: 51)

Clearly it is not to be disregarded that, using our methods, one only learns what the relative pleasantness is with the class of people from which one took the subjects; and of course it is neither to be claimed nor to be expected that that which appeals to educated adult Europeans appeals to children or native Africans – indeed not even that the same appeals to women as appeals to men. However, it does not hurt to pursue the differences that occur between various classes of people according to age, sex, race, social class, level of education, climate, and era, in that one categorizes the subjects accordingly, rather this in itself must also be considered to be an important task of experimental aesthetics. (G. T. Fechner, *Zur experimentalen Aesthetik*: 52)

Up until now, we have generally only discussed the measure of aesthetics solely from an objective perspective, but the previous leads us to attribute a subjective significance to it as well, as it gives us reference itself for the quantitative comparison of the taste of various subjects.

In general, we speak of differing taste, insofar differing things please or displease differing subjects, or also only pleases and displeases to differing degrees. If we consider the taste which prefers the standard ratio to be the standard taste in regard to the simplest ratios – that with the predominant number of votes of educated subjects – we will as such be able to judge and determine the less good or correct taste in this regard according to its greater or lesser deviation from the preferred ratio from the standard ratio, so that when someone does not prove good or correct taste in regard to the simplest cases, it will be even less the case in more complicated cases, where in contrast when it is proved in such, there is indeed no certainty, but a practical prerequisite for the latter will also be present. Accordingly, simple rectangles, crosses etc. or the simplest applications of such forms can be constructed for this as test objects for taste. Those who deviate in my experiments from the standard value of the rectangle, the standard ratios of the cross, always arouse a suspicion to me that

they do not have any good taste in general in purely aesthetic matters. Clearly, the judgment according to this requires the greatest caution; taste could be good in the direct respect as well, but be bad in regard to associative influences. And in any case, the judgment of taste of an individual's single decision is not to be given too much sway because of possible random influence. The described methods would also indeed need be retained, but to be arranged differently; however, I have not yet developed them in this regard. (G. T. Fechner, *Zur experimentalen Aesthetik*: 54)

A great difficulty, however, lies in our method in the requirement of having a large number of individual ascertainments at our disposal, because the opportunity to obtain many ascertainments is always more or less limited. And I have also already mentioned that quite a few of my series of tests therefore remained incomplete and many important ascertainments that could have been settled here are not yet. The circle of acquaintances that one finds the opportunity to consult for the methods of choice and creation is soon exhausted, and the cases for utilization do not in themselves present themselves in sufficient numbers under the conditions suitable for the measures. Therefore, for a complete implementation of the experiments according to our methods, multiple people would have to unite in order to utilize the opportunities at their disposal and to create such themselves according to a common plan. I have already paid tribute to the support in this regard that I received from several young friends in the preface. (G. T. Fechner, *Zur experimentalen Aesthetik*: 67–68)

In conclusion, with aesthetic laws it is the same as with laws of nature. It cannot be proved through any observations, experiments or calculations that the law of gravity corresponds *exactly* to the inverse ratio of the square of the distance. It must suffice to furnish proof with such approximation that the deviations seem to be explicable on account of the instability and disturbances of the observations. And the corresponding must be valid for the proof of the legitimacy of the golden ratio or any other ratio as an aesthetic standard ratio. The observations in this case, just as in the other, only lead to numerical ratios that are very close to those to which one adheres as truly lawful or standard; and in the end one only adheres to these because one finds a greater satisfaction in attributing fundamental laws to ratios that allow for simple expression, or that can be created independently from a simple principle, than seemingly motiveless deviations from it. (G. T. Fechner, *Zur experimentalen Aesthetik*: 81)

B.2.3 *Vorschule der Aesthetik* (1876)

The twofold manner in which human understanding strives to substantiate and to develop itself is also exercised in aesthetics – the study of pleasure and displeasure, or, according to others, the study of beauty. One approaches it, in short, *from top* to bottom, in which one descends from the most general ideas and terms going into the individual, or *from bottom* to top, in which one ascends from the individual to the general. *In the former*, one categorizes and subcategorizes the field of aesthetic experience into an ideal constructed framework from the uppermost perspectives; *in*

the latter,[19] one constructs the entire field of aesthetics based on aesthetic facts and laws from the bottom up. *In the former*, first and foremost, it is about the ideas and concepts of beauty, art, style, about its place in the system of the most general concepts – especially its relation to what is true and good; and one gladly ascends with such to the absolute, the divine, divine ideas and divine creator's activities. From the sheer loftiness of such generalities, one then descends into the material, emperical domain of the individual, temporally and spatially beautiful, and measures everything individual on the scale of the general. *In the latter*, one begins with lessons learned about what pleases and what displeases, supports all concepts and laws that have their place in aesthetics on this, seeks to generalize them more and more with consideration for the general laws of obligation, to which those of liking must always remain subordinate, and through such means to arrive at a system of the most general concepts and laws. (G. T. Fechner, *Vorschule der Aesthetik*, First Part: 1)

In accordance with us liking or disliking that which we remember about something, the memory also contributes a factor of like or dislike to the aesthetic impression of that thing, which can work harmoniously or in conflict with other moments of memory and the direct impression of that thing, from which the most multifarious aesthetic relations arise, which we have already had an opportunity and will again find the chance to delve into further. The strongest and most frequent effects that we experience from something, in connection with something, and in comparison with something, of course also leave memories that most powerfully intervene in the associative impression. (G. T. Fechner, *Vorschule der Aesthetik*, First Part: 94)

Through aesthetic experiments, however, a clear answer to these questions was attainable that does not correspond completely to Wolff, nor Heigelin, nor Zeising. In order to summarize the results beforehand – of which indeed not all but most emerge in the experiment to be discussed in the following – they are as follows. Regarding the others one is meanwhile referred to the future continuation of the writing already mentioned several times.

a) Among all rectangular forms, on the one hand the square with rectangles closest to it and on the other hand the very long rectangles are the most unpleasant.

b) The square itself seems to be surpassed in pleasantness by the rectangles closest to it, or at most has a doubtful advantage over them.

c) The simple rational ratios that correspond to the frequency ratios of musical consonances have no advantage in pleasantness at all as aspect ratios of rectangles over the dissonant musical ratios that can be expressed in less small numbers.

d) The rectangle formed according to the golden ratio together with those closest to it indeed has an advantage in pleasantness over the remaining rectangles.

e) A slight deviation of any rectangle from symmetry, however, does much more damage to its pleasantness than a comparatively much more significant deviation

[19]The translation of "*hier*" and "*dort*" in this part presented difficulties: "*hier*" [translated: in the latter] also refers to this book and "*dort*" [translated: in the former] is referring to previous "top-down" systems.

from the ratio of the golden section, and in general its advantage disproportionately less decisive and traceable than that of symmetry.

f) In regards to the division of a horizontal length (parallel to the eyes), the golden ratio has a decisive disadvantage in comparison to an even division, in which we have another example of the case noted on pp. 63 and 65 that under certain circumstances, by ascending to a higher level of homogeneity, the loss incurred by thus damaging the lower level of homogeneity cannot be compensated for.

g) In regards to the division of a vertical length (or, more generally, perpendicular to the eyes), according to experiments in completing crosses, the most advantageous division of the longer beam changes according to the ratios of the cross-beam. For the most favorable ratio of the cross-beam to the longer beam, however, the division according to the golden ratio is not the most advantageous, but rather that according to the ratio of 1:2 of the shorter to the longer part.

Accordingly, I cannot refrain from finding the aesthetic value of the golden ratio to be overestimated by Zeising, in which I indeed do not deny the interest and accomplishment of the Zeisingian discovery that this ratio has a notable aesthetic value in general – indeed expressly see a *discovery* in such. I also do not want to deny, considering that my experiments are by far not expansive enough to come to a generally discounting judgment in this regard, that under certain circumstances, which, however, would first have to be investigated and formulated more exactly, that the golden ratio could prove to have an advantage as a divisional ratio – namely probably when a length divided according to the golden ratio is combined symmetrically with a different length. Certain is only that the aesthetic advantage of the golden ratio is not to be assumed as simply as it is presented to us by Zeising. (G. T. Fechner, *Vorschule der Aesthetik*, First Part: 191–193)

Most explained from the start that, according to the utilization, either one or another rectangle could be the most pleasant. I admitted this, but continued, asking whether they could not indeed find one of these rectangles more pleasant, satisfying, harmonious, elegant, than another by virtue of its different aspect ratio apart from any consideration of purpose and significance, and which they would prefer as the most pleasant form *on average*. Then three cases were possible. Either all, or the greater majority, refused a judgment after this, because no difference was to be found, or judgments were indeed made that were nonetheless scattered indifferently between the preference and rejection of the various rectangles; or it showed itself to be true after counting up all of the cases that a certain type of rectangle had such a large majority in numbers of preferences – that in general such order emerged in the succession of preferences – that this could not be attributed to randomness.

The success was this: only in very few cases was a judgment completely refused; but also in very few cases was the judgment extremely decisive and certain, although there were a few such. Usually a longer wavering took place, and when one had finally decided on a rectangle, one preferred another one afterwards sometimes in the same experiment, in a self-correcting manner, or remained wavering between two, three or even four rectangles. If the experiment was repeated at a different time with the same people after the impression of the earlier experiment was erased, as happened a few

times, a different rectangle was not rarely preferred, which was neighboring in terms of its ratio, instead of the rectangle preferred in the earlier experiment. Despite this uncertainty in particular, the above table shows very decisive results on the whole. (G. T. Fechner, *Vorschule der Aesthetik*, First Part: 195–196)

If one presents young children with only both forms ⊙ and ▫ with the same surface area on nicely colored paper as children love, not with the question which they like the best, but with the permission to take one of them, then they clumsily grab one or the other without there seemingly being a difference to them, and without there being a significant majority of the Z on one side in the end. This turned out to be the case in experiments that I had performed in two children's day care centers, observing the caution of changing the position of the ⊙ and ▫ in terms of right and left for the various children – so that the preferential grasping with the right hand does not make a difference – and of laying down ⊙ just as often parallel to the eyes as perpendicular.

Indisputably, the ratios of pleasantness of the various forms of rectangles must also be proved in applications; only that partially the purpose and partially combinatorial influences have a multilaterally altering effect. Indeed one finds, however, insofar as no opposing effects from such influences result, that the golden ratio and those rectangles close to it are preferred. In contrast, the long rectangles and the square with the rectangles close to them are unpopular. Raw appearance even makes this clear. In addition to this, however, I have carried out many measurements on entire classes of rectangular objects that occur in doings and dealings, which confirm this, which I will nonetheless not get into more detail about here, in order to merely mention the following results in general briefly. (G. T. Fechner, *Vorschule der Aesthetik*, First Part: 199)

The best taste is that with which on the whole the best for humanity emerges; the best for humanity is, however, that which is more in the interest of its foreseeably eternal well-being over time. (G. T. Fechner, *Vorschule der Aesthetik*, First Part: 264)

C Minor Modifications to the Demographic Data Collected

Some small changes concerning the survey portion had to be made to the .txt files created by the program used in the experiment prior to inputting the information into the database. The following is a summary of the changes made to the raw data collected concerning the subjects' majors or programs:

- Some subjects included an emphasis in their major whereas others did not. Emphases were removed in order to group subjects solely according to major.
- "CCS Literature" was changed to "Literature."
- "BS," "BM" or "BA" were removed several times.
- "Undecided" was changed to "Undeclared."
- "Bus Econ Accounting" was unclear in one subject's file. This was changed to "Business Economics," as accounting was assumed to be an emphasis.
- "Lecturer" was once changed to "Media Arts and Technology," because the experimenter was familiar with the subject.
- "MAT staff" was changed to "N/A."
- One MAT faculty member had indicated N/A, so this was changed to "Media Arts and Technology."
- Some subjects used abbreviations, and these were all unabbreviated.

Regarding the country of birth and nationalities of the subjects, the following information was modified:

- States of the USA were changed to "USA."
- "Hong Kong" was once changed to "China."
- A large number of subjects misunderstood the meaning of nationality (e.g. a subject claimed to be Greek although he was born in the USA), perhaps confusing nationality with their ancestor's national or ethnic origin. If a person was born in the USA, their nationality was changed to "USA."
- Two subjects with whom the experiment was familiar had dual citizenship, and the experimenter decided for the sake of simplicity to attribute to the subjects the countries with which they seemed to most clearly associate themselves.

Regarding the questions about subjects' familiarity with the experimenter's work, and their level of musical and mathematical education, the following changes were undertaken:

- Paid participants' claims that it was "true" that they had discussed the experimenter's work with him were changed to "false," as the experimenter knew none of them personally and had never conversed with them before their involvement in the experiment.
- Three pair participants claimed to have seen the experimenter's website, which seemed highly unlikely, so this was also changed to "false."
- One 19-year-old subject claimed to have an undergraduate degree in music, and this was changed to "false."
- One 20-year-old subject claimed to have an undergraduate degree in math, and this was changed to "false."

No changes were made to the sonifications the subjects chose.

D A Table of the Results Obtained in the Experiment

The table contained in this appendix is an abridged version of the more comprehensive table available at http://caseymongoven.com/experiment/I. 165 subjects are included in the table. The numbers in the columns denoted P1 through P5 (pair 1 through pair 5) reflect the pair index, or how many times a Fibonacci-related sonification was chosen. See 7.1 for a description of the pairs. As discussed in 7.3.2, the maximum value for the pair index (PI) is 3 and the minimum 0. The following is a key to the headings in the table:

P1 = binary sequences
P2 = fractional part of integer multiples
P3 = numeral systems
P4 = signature sequences
P5 = irrational numbers
FI = Fibonacci index (see 7.3.1)
SFI = secondary Fibonacci index (see 7.3.1)
CI = consistency index (see 7.3.3)

subject	P1	P2	P3	P4	P5	FI	SFI	CI	age	sex
1	1	1	1	2	1	6	0	0%	19	F
2	3	2	0	2	1	8	0	40%	19	F
3	1	2	0	1	1	5	-1	20%	20	F
4	3	1	2	0	2	8	0	40%	21	F
5	3	2	1	2	3	11	2	40%	21	F
6	2	1	0	3	0	6	-1	60%	21	F
7	0	1	0	1	2	4	-2	40%	19	F
8	3	3	1	3	1	11	3	60%	20	M
9	2	1	3	1	0	7	0	40%	21	F
10	1	1	0	2	2	6	-1	20%	18	F
11	3	2	3	2	1	11	2	40%	18	F
12	3	2	0	2	1	8	0	40%	20	M
13	0	1	0	1	1	3	-2	40%	20	F
14	2	1	1	1	1	6	0	0%	19	F
15	2	2	1	1	1	7	0	0%	20	F
16	3	3	1	2	2	11	2	40%	20	F
17	1	2	2	0	2	7	-1	20%	19	M
18	1	2	3	0	0	6	-1	60%	22	M
19	0	2	2	0	0	4	-3	60%	19	F
21	0	1	0	1	0	2	-3	60%	19	M
22	3	2	0	0	2	7	-1	60%	20	M
23	0	0	0	2	3	5	-2	80%	22	F
24	2	2	1	1	1	7	0	0%	19	F
25	2	2	0	1	2	7	-1	20%	19	F
26	3	1	1	1	2	8	1	20%	21	F
27	0	3	0	2	2	7	-1	60%	18	F
28	1	2	0	0	2	5	-2	40%	18	M
29	3	0	2	2	2	9	0	40%	20	F
30	0	1	1	1	2	5	-1	20%	18	M
31	2	2	3	2	2	11	1	20%	18	M
32	1	2	0	2	0	5	-2	40%	18	F
33	3	3	0	1	1	8	1	60%	19	M
34	1	2	2	1	1	7	0	0%	19	F
35	2	1	3	0	0	6	-1	60%	22	M
37	1	3	3	1	2	10	2	40%	19	F
38	1	0	3	2	1	7	0	40%	18	F
39	3	3	1	2	1	10	2	40%	20	F
40	3	0	1	1	3	8	1	60%	19	M
41	0	1	1	0	2	4	-2	40%	21	M
42	1	1	0	0	1	3	-2	40%	19	M
43	3	2	2	3	0	10	1	60%	21	F
44	3	3	0	2	0	8	0	80%	20	F

45	1	3	3	1	1	9	2	40%	20	F
46	3	1	1	2	3	10	2	40%	19	F
47	0	3	2	0	2	7	-1	60%	18	M
48	2	1	2	1	2	8	0	0%	20	M
49	1	2	2	2	1	8	0	0%	20	M
50	2	0	1	1	2	6	-1	20%	18	M
51	1	1	2	1	1	6	0	0%	21	F
52	2	2	1	1	1	7	0	0%	21	F
54	2	1	3	0	2	8	0	40%	20	F
55	2	2	0	0	0	4	-3	60%	20	F
56	2	2	2	1	1	8	0	0%	21	M
57	1	1	0	2	2	6	-1	20%	21	F
58	1	0	3	1	1	6	0	40%	18	F
59	2	2	3	2	1	10	1	20%	22	F
60	2	3	2	2	1	10	1	20%	18	F
61	3	2	2	2	0	9	0	40%	18	F
62	1	3	1	2	2	9	1	20%	19	F
63	1	3	1	3	1	9	2	40%	18	F
64	2	0	0	1	2	5	-2	40%	18	F
65	1	2	1	1	1	6	0	0%	21	F
66	3	3	1	1	2	10	2	40%	18	F
67	2	1	1	2	3	9	1	20%	22	M
69	0	2	0	0	2	4	-3	60%	18	F
70	2	1	1	2	3	9	1	20%	21	F
71	2	2	0	3	2	9	0	40%	19	F
72	1	3	1	1	1	7	1	20%	21	F
73	1	0	1	1	2	5	-1	20%	19	F
74	1	1	0	2	1	5	-1	20%	21	M
75	1	2	2	1	2	8	0	0%	18	M
76	3	0	1	1	1	6	0	40%	20	F
77	0	0	3	0	2	5	-2	80%	20	F
78	3	0	2	1	2	8	0	40%	21	F
79	1	2	1	1	1	6	0	0%	21	F
80	1	1	1	0	3	6	0	40%	20	F
81	1	1	2	3	2	9	1	20%	19	F
82	2	3	3	0	1	9	1	60%	19	F
83	2	2	1	0	1	6	-1	20%	21	F
84	1	1	1	1	1	5	0	0%	20	M
85	3	1	2	2	1	9	1	20%	21	F
86	2	1	3	0	2	8	0	40%	19	F
87	2	0	1	1	1	5	-1	20%	22	F
88	1	1	1	0	2	5	-1	20%	21	F
89	2	1	2	1	2	8	0	0%	19	F
90	3	0	2	0	2	7	-1	60%	21	M
91	1	3	2	2	2	10	1	20%	22	F
92	2	3	3	3	2	13	3	60%	22	F
93	1	3	1	0	1	6	0	40%	20	M
94	2	1	3	1	1	8	1	20%	20	F
95	2	3	2	1	2	10	1	20%	19	F
96	1	1	3	1	0	6	0	40%	20	F
97	2	2	0	1	2	7	-1	20%	21	M
98	0	2	1	2	2	7	-1	20%	20	F
100	0	2	0	2	2	6	-2	40%	22	F
101	0	2	2	2	2	8	-1	20%	32	M
102	0	1	2	1	2	6	-1	20%	29	M
103	1	0	2	2	1	6	-1	20%	36	M
104	1	2	0	2	1	6	-1	20%	35	M
105	0	0	2	3	1	6	-1	60%	33	M
106	2	0	0	3	3	8	0	80%	30	M
107	0	0	1	2	0	3	-3	60%	61	M
108	2	0	1	1	0	4	-2	40%	29	M
109	1	0	3	1	1	6	0	40%	67	M
110	3	2	0	2	1	8	1	40%	41	M
111	3	3	3	2	2	13	0	60%	53	M

112	1	0	0	2	1	4	0	40%	24	M
113	3	0	2	1	1	7	0	40%	19	F
114	2	0	2	3	3	10	1	60%	20	M
115	1	2	0	2	3	8	0	40%	18	F
116	2	2	1	2	1	8	0	0%	21	F
117	3	1	0	2	2	8	0	40%	18	F
118	2	3	2	1	2	10	1	20%	19	F
119	0	0	3	1	3	7	0	80%	43	M
120	2	2	0	1	2	7	-1	20%	19	M
121	1	1	1	3	1	7	1	20%	22	F
122	1	2	1	2	1	7	0	0%	20	F
123	3	3	1	3	1	11	3	60%	20	F
124	2	0	3	0	3	8	0	80%	19	F
125	2	3	2	0	3	10	1	60%	22	F
126	3	2	1	1	0	7	0	40%	21	M
127	2	0	1	2	3	8	0	40%	18	F
128	2	3	0	1	0	6	-1	60%	19	F
129	3	3	1	0	2	9	1	60%	20	M
130	2	3	0	1	2	8	0	40%	20	F
131	3	2	1	1	3	10	2	40%	39	F
132	2	0	2	1	2	7	-1	20%	18	F
133	3	1	1	1	2	8	1	20%	18	M
134	3	2	1	1	1	8	1	20%	20	F
135	1	0	0	0	0	1	-4	80%	18	M
136	2	3	1	2	2	10	1	20%	18	F
137	1	2	0	0	2	5	-2	40%	19	F
138	1	1	1	1	3	7	1	20%	22	F
139	1	0	0	0	0	1	-4	80%	18	F
140	1	0	0	3	2	6	-1	60%	19	M
141	2	1	2	1	2	8	0	0%	18	F
142	1	2	3	1	2	9	1	20%	19	M
143	0	2	2	2	2	8	-1	20%	18	M
144	2	1	1	0	3	7	0	40%	18	M
145	0	1	1	1	1	4	-1	20%	19	F
146	3	3	0	2	2	10	1	60%	20	F
147	0	2	0	0	3	5	-2	80%	18	F
148	1	1	3	2	0	7	0	40%	29	F
149	3	0	0	2	1	6	-1	60%	20	M
150	2	1	2	2	1	8	0	0%	18	F
151	1	2	0	0	2	5	-2	40%	22	F
152	0	1	2	3	3	9	1	60%	53	M
153	1	1	2	0	1	5	-1	20%	19	F
154	2	1	2	0	3	8	0	40%	20	F
155	3	2	0	2	1	8	0	40%	21	M
156	0	0	1	3	2	6	-1	60%	24	M
157	1	3	3	0	1	8	1	60%	38	M
158	1	0	3	0	2	6	-1	60%	37	M
159	1	2	2	1	3	9	1	20%	63	M
160	0	1	1	1	2	5	-1	20%	45	M
161	0	1	1	1	2	5	-1	20%	36	F
162	0	1	0	1	3	5	-1	60%	51	F
163	1	2	0	1	3	7	0	40%	35	F
164	1	1	2	2	0	6	-1	20%	42	M
165	2	2	0	0	1	5	-2	40%	49	F
166	0	2	0	2	2	6	-2	40%	32	F
167	1	2	0	2	3	8	0	40%	23	M
168	3	0	1	1	1	6	0	40%	41	M
169	3	1	1	2	1	8	1	20%	35	F
170	2	1	0	1	0	4	-2	40%	25	F
averages	**1.58**	**1.47**	**1.24**	**1.29**	**1.55**	**7.12**	**-0.21**	**35.8%**	**23.3**	**F 63.6% M 36.4%**

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