AUTHOR: Mongoven, Casey, P.
TITLE: Trends in the Use of the Fibonacci Sequence in Musical Compositions
KEYWORDS: golden ratio, Fibonacci, analysis, trends, twentieth-century
ABSTRACT: Many composers of the twentieth and twenty-first centuries have used the Fibonacci sequence and the golden ratio in their work. Certain trends have emerged in their utilization that have not yet been discussed in the literature. After a brief introduction to the Fibonacci sequence and the golden ratio, this article begins by developing a framework for analysis in which various potential applications of the Fibonacci sequence are presented. It then commences with an analytical survey of various works by 25 different composers who have used the Fibonacci sequence or the golden ratio in their work. Trends are then identified and summarized in a table.

Casey P. Mongoven
Department of Germanic, Slavic & Semitic Studies
4206 Phelps Hall
University of California, Santa Barbara
Santa Barbara, CA 93106-4130

ACCOMPANYING FILES: none

Trends in the Use of the Fibonacci Sequence in Musical Compositions

February 8, 2014

Abstract

Many composers of the twentieth and twenty-first centuries have used the Fibonacci sequence and the golden ratio in their work. Certain trends have emerged in their utilization that have not yet been discussed in the literature. After a brief introduction to the Fibonacci sequence and the golden ratio, this article begins by developing a framework for analysis in which various potential applications of the Fibonacci sequence are presented. It then commences with an analytical survey of various works by 25 different composers who have used the Fibonacci sequence or the golden ratio in their work. Trends are then identified and summarized in a table.

1 The Golden Ratio and the Fibonacci Sequence

We define the division of a line segment AC to be by the golden ratio (henceforth GR) when we divide it at the point B such that the resulting line segments have the property BC:AB = AB:AC.



GR is thus defined to be a pair of inversely related values: $\varphi = \frac{\sqrt{5}-1}{2}$ and $\Phi = \frac{\sqrt{5}+1}{2}$. Euclid provided the first known definition of GR in his *Elements* (c. 300 BCE). GR is also sometimes referred to as the *golden section*, *golden number*, or *golden mean*.

In addition to the standard Fibonacci sequence (henceforth FS) $F_n = < 0, 1, 1, 2, 3, 5, 8, 13, \cdots >$, in which $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 0$ and $F_1 = 1$, many composers have used generalized Fibonacci sequences as well. We define a generalized Fibonacci sequence (henceforth GFS) G_n to begin with two integers $G_0 = a$ and $G_1 = b$. Values for $0 > n > 1$ are given by the Fibonacci rule $G_n = G_{n-1} + G_{n-2}$. For example, one G_n that has been the subject of significant mathematical research is the Lucas sequence (henceforth LS) $L_n = < 2, 1, 3, 4, 7, 11, 18, \cdots >$, where $G_0 = 2$ and $G_1 = 1$. One can see the relation of FS to GFS in the following table:

n	0	1	2	3	4	5	6	7	...
G_n	a	b	$a+b$	$a+2b$	$2a+3b$	$3a+5b$	$5a+8b$	$8a+13b$	...

Just as in F_n, where $G_0 = 0$ and $G_1 = 1$, the ratio of two successive terms of any G_n where $G_0 \neq 0$ and $G_1 \neq 0$ converges to GR as n approaches infinity [1]:

$$\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \Phi$$

We will also occasionally see composers make use of the inverse G_n sequence, in which $G_n = G_{n-2} - G_{n-1}$.

FS was known in India at least several centuries prior to Fibonacci's (aka Leonardo Pisano) famous rabbit problem in 1202. A method for its formation was given by Virahāṅka (c. 600–800 CE) [2].

2 Applications of GFS to Musical Parameters

Innumerable composers of the twentieth and twenty-first centuries have applied GFS or GR in their compositions in various ways, and certain trends have emerged in their application. Before preceding with an analytical survey of works, it will be helpful to develop a framework for analysis in which various potential applications are presented.

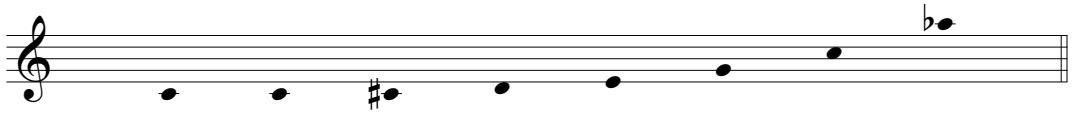
Although many composers have applied GFS or GR in their compositions, a direct application of GR is rare. This can perhaps be attributed to the fact that GR—whether applied to pitch, rhythm, meter, etc.—generally cannot be realized with great accuracy by performers on traditional instruments. Xenakis once wrote [3]:

> The golden number is irrational in mathematics. In music it is not possible (exception: electronic music). Providing durational values that a normal musician can count, I formed a [Fibonacci] sequence as just described.

We will therefore concentrate on applications of GFS in developing our framework, as they will prove most useful in analysis. Applications that do not fit into this framework can be considered anomalies and particularly unique. In developing this framework, it shall not be our purpose to exhaust all possible applications, but rather to shed light onto those that are most common as well as those that seem most intuitive. We will see that, in practice, these applications are utilized together with various other compositional techniques that have nothing to do with GFS or GR.

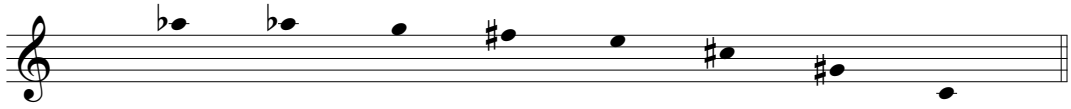
2.1 Pitch

We can create a sequence of pitches—or equivalently, a sequence of intervals—based on FS in twelve-tone equal temperament; these are called *Fibonacci pitch sequences*:



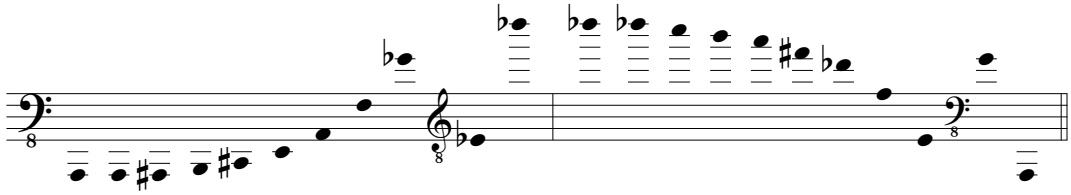
interval:		0	1	1	2	3	5	8
pitch:	1	1	2	3	5	8	13	21

This could be inverted, and we could start on a different pitch:

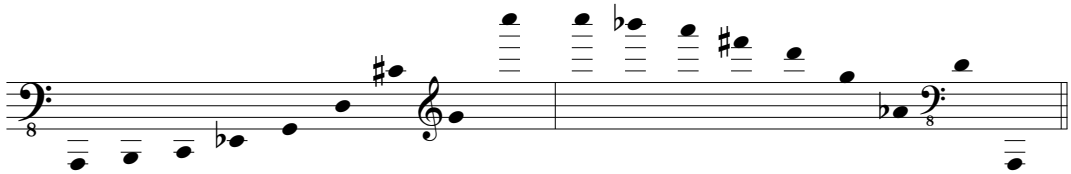


interval:	0	1	1	2	3	5	8
pitch:	1	1	2	3	5	8	13	21

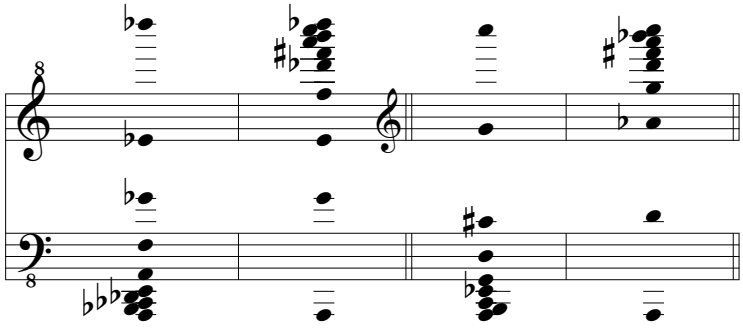
The primary challenge composers seem to face is that the pitches very quickly exceed the range of our hearing; this issue is not unique to any particular application and is due to the additive nature of FS. Even if we started with extremely small intervals—and even if it took a while for that reason—as soon as our ear could discriminate an interval, from that point it would very quickly escape our hearing range. The following example shows the largest possible audible sequence of intervals based on FS in twelve-tone equal temperament $< 0, 1, 1, 2, 3, 5, 8, 13, 21, 34 >$:



If we utilize LS, the largest possible sequence of intervals is $< 2, 1, 3, 4, 7, 11, 18, 29 >$:



The same sequences can be expressed harmonically:



Some composers have used Fibonacci sequences under a modulus, creating what are called *Fibonacci pitch class sequences*. In this case, $X_n \equiv X_{n-1} + X_{n-2} \pmod{m}$.

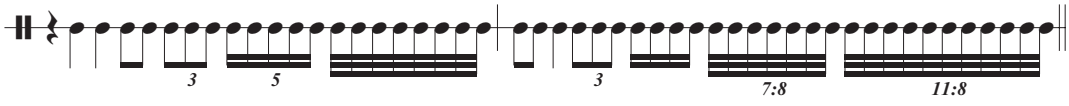
With a modulus we lose the property that each interval is the sum of the previous two, but are free to take advantage of octave equivalence, and the property that each interval or pitch class is the sum of the previous two is retained. If we exhaust all possible GFS sequences in twelve-tone equal temperament, the result is a series of ten periodic sequences. In the following table, the sequences have been truncated to the length of their period:

0																							
0	1	1	2	3	5	8	1	9	10	7	5	0	5	5	10	3	1	4	5	9	2	11	1
0	11	11	10	9	7	4	11	3	2	5	7	0	7	7	2	9	11	8	7	3	10	1	11
0	2	2	4	6	10	4	2	6	8	2	10	0	10	10	8	6	2	8	10	6	4	10	2
0	3	3	6	9	3																		
0	9	9	6	3	9																		
0	4	4	8	0	8	8	4																
0	6	6																					
2	1	3	4	7	11	6	5	11	4	3	7	10	5	3	8	11	7	6	1	7	8	3	11
10	11	9	8	5	1	6	7	1	8	9	5	2	7	9	4	1	5	6	11	5	4	9	1

This is sometimes referred to as a complete Fibonacci system modulo 12 [4]. Every possible ordered interval class and pitch class dyad is contained once and only once in the system. In other words, any two integers between 0 and 11 will be found next to one another in this table once and only once.[1] In addition to application to pitch, such sequences could be treated in a serial manner and applied to various other musical parameters as well.

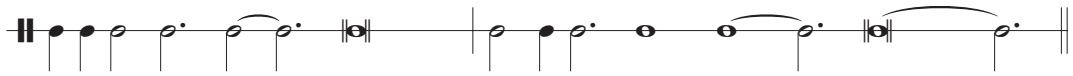
2.2 Rhythm

Some composers have applied GFS to rhythm. Two general possibilities exist, the first of which is called *Fibonacci attack summation*, in which, given two starting values, each beat contains a number of equally spaced attacks equal to the number of attacks that occurred in the previous two beats. $< 0, 1, 1, 2, 3, 5, 8 >$ and $< 2, 1, 3, 4, 7, 11 >$ could be utilized, for example, as such:



[1] In some cases (e.g. $< 1, 0 >$ or $< 11, 2 >$), due to periodicity and the truncation in this table, it will be necessary to wrap around in the table from the last integer in a sequence to the first.

The second technique, called *Fibonacci pulse summation*, is in a sense the inverse of the above: each attack contains a number of pulses equal to the number of pulses attributed to the previous two attacks. Put another way, each note value is equal to the sum of the previous two. The latter definition might be more relevant for sustaining instruments, the former for those with a shorter attack. $< 0, 1, 1, 2, 3, 5, 8 >$ and $< 2, 1, 3, 4, 7, 11 >$ could be utilized, for example, as such:



2.3 Meter

Another temporal application of GFS directly related to Fibonacci pulse summation is *metric summation*, in which, given two starting values, each measure contains the number of beats contained in the previous two. We could create a series of meters based on FS, for example, as such:

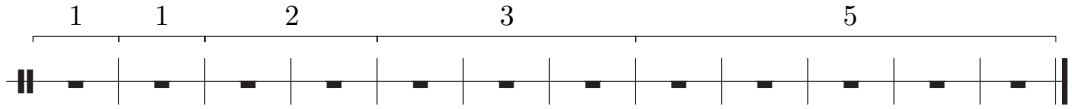


LS could be utilized as well:

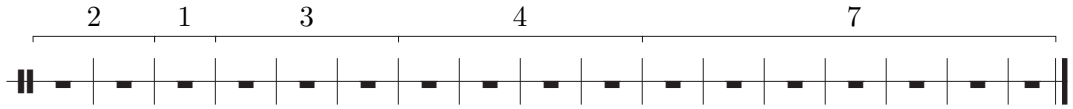


2.4 Proportion and Form

One can design formal proportions in a composition according to FS, so that the lengths of sections are defined according to FS, as such:

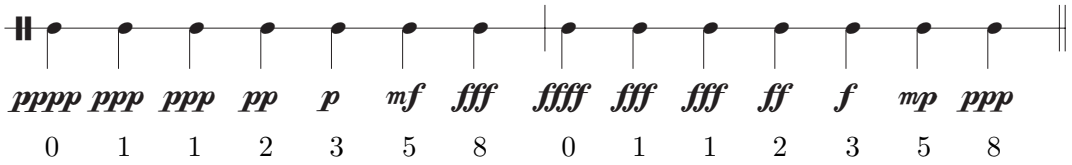


Or according to LS:

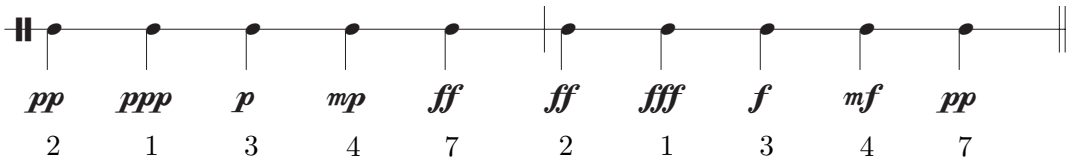


2.5 Dynamics

GFS could also be applied to dynamics. For example, $< 0, 1, 1, 2, 3, 5, 8 >$ could be applied to dynamics in the following two ways, either increasing in loudness, or decreasing:

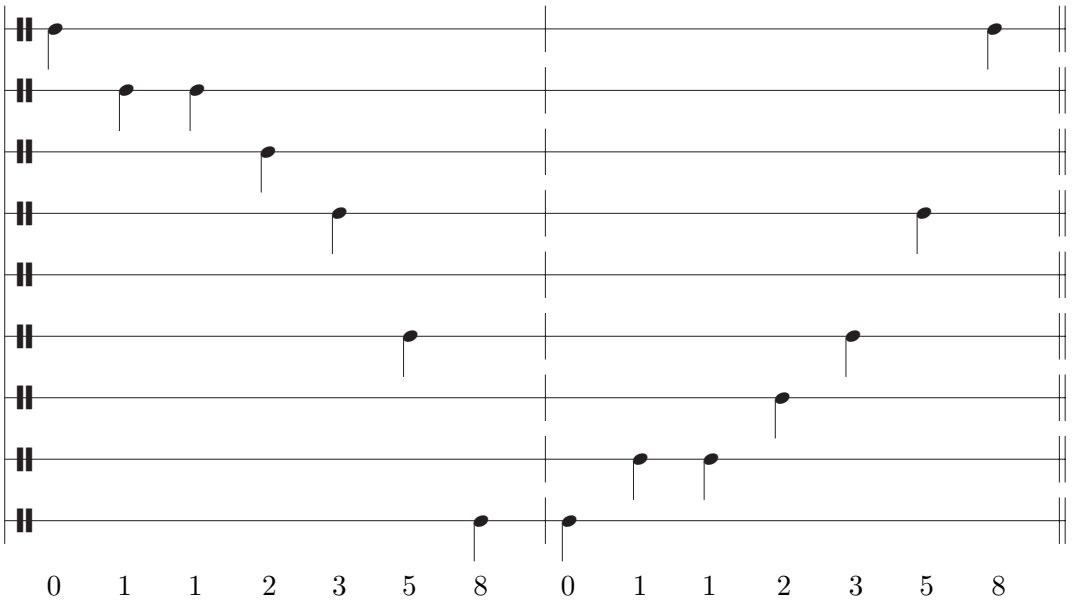


LS $< 2, 1, 3, 4, 7 >$ might appear as such:



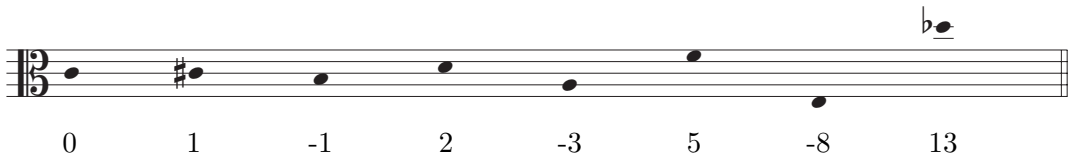
2.6 Spatial Location

Mapping GFS to spatial location is possible as well; $< 0, 1, 1, 2, 3, 5, 8 >$ might emanate from different locations. If nine players (representing the integers 0 through 8) were lined up equidistantly, FS could be applied in the following manner:



2.7 Inverse GFS

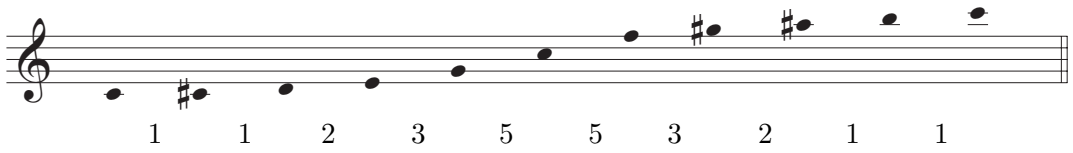
As mentioned earlier, an inverse GFS is a sequence in which $G_n = G_{n-2} - G_{n-1}$. If the starting values are $G_0 = 0$ and $G_1 = 1$, as in the standard FS, then the sequence begins $< 0, 1, -1, 2, -3, 5, -8, 13, \cdots >$. This sequence could be applied to pitch, for example, as follows:



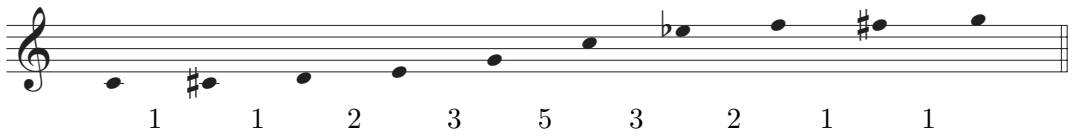
Applying negative numbers in the time domain does not make sense for our purposes, but we could potentially apply inverse GFS to other parameters such as dynamics and location as well in a similar manner as has been demonstrated.

2.8 Reversal

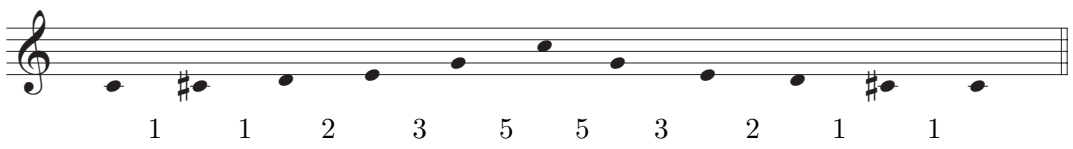
Perhaps due to the property of GFS mentioned earlier that the integers in the sequence rapidly increase, many composers have used a technique called *reversal* in which, at a certain point, they reverse their direction in the sequence (in turn utilizing an inverse GFS). Reversal can take different forms depending on the musical parameter GFS is being applied to. In the pitch domain, several possibilities exist. The first is called *reversal with repetition*, because the pivot interval (in this case 5) is repeated:



The second, *reversal without repetition*, might take the following form:

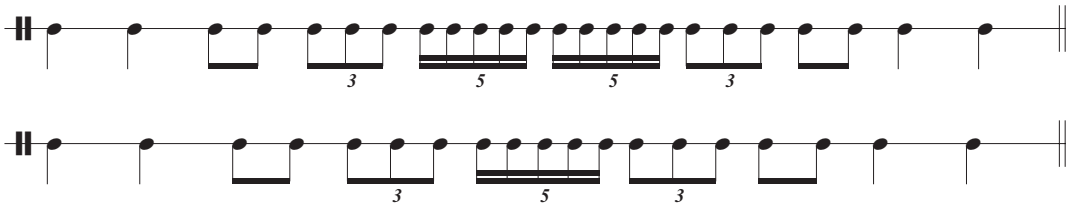


In the pitch domain, it is also possible to simply switch the direction from ascending to descending, or vice versa. This is called *reversal with change of direction*:

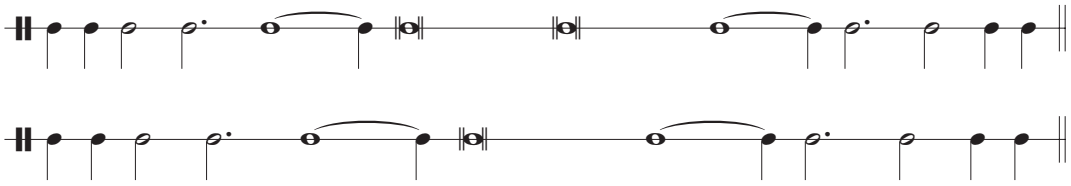


In the temporal domain, no change of direction is possible: we cannot go backwards in time. There are thus two possibilities for reversal in metric and rhythmic applications: reversal with repetition and reversal without repetition.

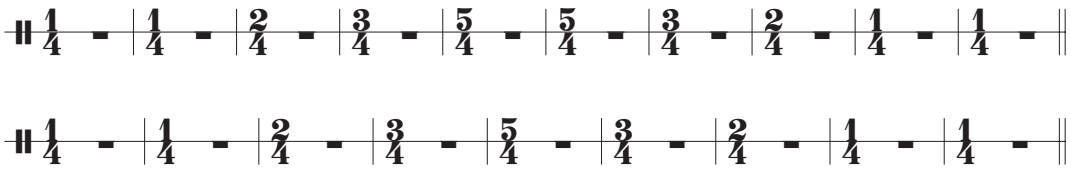
Reversal with attack summation, with and without repetition:



Reversal with pulse summation, with and without repetition:



Reversal in the metric domain, with and without repetition:



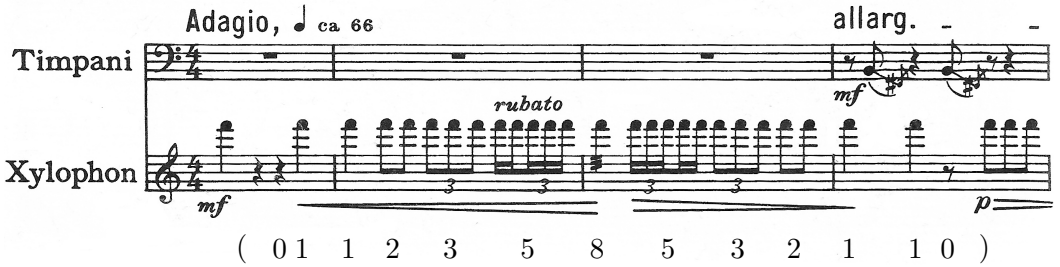
Reversal can be utilized in almost any domain GFS can be applied.

3 A Survey of Works Utilizing GFS and GR

We will now undertake brief analyses of the works of a number of different composers. Our primary goal will be to gain some insight into the variety of ways in which GFS and GR have been applied and to identify trends that have emerged. We will proceed in chronological order. In order to avoid making whether or not the composer utilized GFS and GR the subject of these analyses, we will only analyze works where it is obvious or known that a conscious application has taken place.

3.1 Bartók's *Music for String Instruments, Percussion and Celesta* (1936)

One of the earliest apparent conscious applications of FS occurs in the third movement of Béla Bartók's *Music for String Instruments, Percussion and Celesta* [5].[2] We could consider the rhythm in the first four measures of the movement to be a form of Fibonacci attack summation with reversal, although the beat in the second measure that contains five attacks does not use a quintuplet, but two sixteenths followed by three triplet sixteenths:



The indication of *rubato* at that point makes it clear that Bartók did not want this rhythm to be executed with perfect mechanical precision. The 0 on the final beat of the pattern (the third beat of the fourth measure) is not represented by a full quarter rest, but by an eighth rest; the tempo, however, has slowed by that point, so that it might be considered to be the final integer of the mirrored pattern. Multiple authors have suggested that Bartók used FS and GR in other works as well [7] [8]. Bartók is not known to have ever discussed why he utilized FS in composition.

3.2 The Contributions of Joseph Schillinger (1941)

Although the author knows of no practical work in which Joseph Schillinger utilized GFS, his theoretical work would be too important not to include in this survey. In *The Schillinger System of Musical Composition* [9], published in 1941 posthumously, he gives examples of GFS applied to pitch with octave transpositions, as seen below:

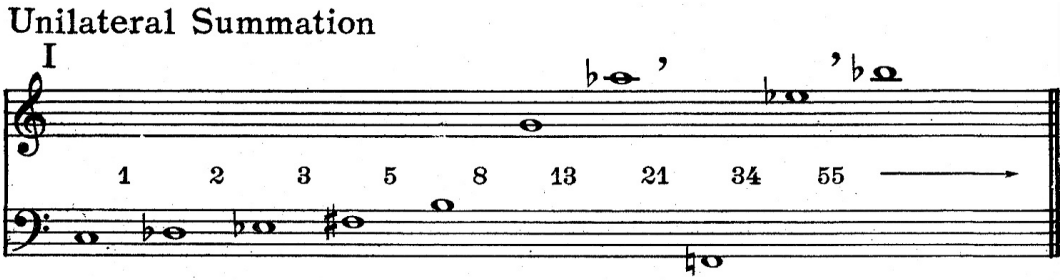


Figure 106. Unilateral symmetry of Fibonacci series (concluded).

[2] Although the first movement of *Music for String Instruments, Percussion and Celesta* has been analyzed by multiple authors in terms of proportions based on FS and GR, Roy Howat has shown that there are significant inaccuracies in such analyses [6].

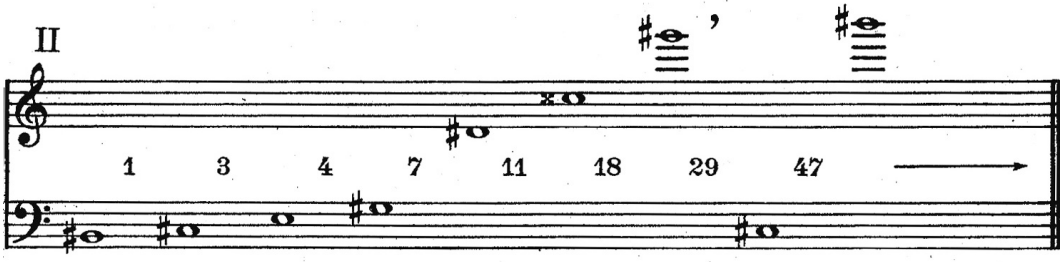


Figure 108. Second summation series applied to melody.

He stresses the symmetric nature of FS:

> A property of the summation series known as Fibonacci series [sic] is that it contains symmetry throughout. The word *symmetry* emphasizes the equality of two measured ratios, according to an authority on the subject, Dr. William Churchill.

He also mentions inverse Fibonacci pitch sequences, which he describes as being characterized by bilateral symmetry.

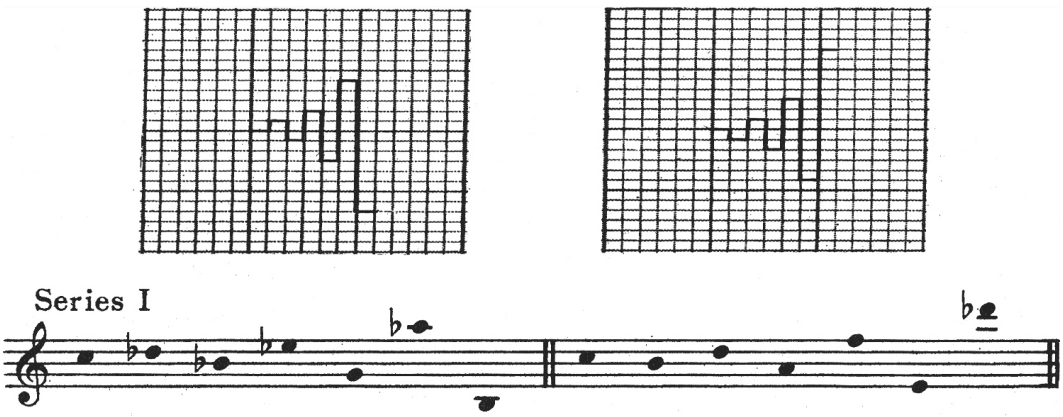


Figure 112. Bilateral symmetry in first summation series.

We note that his notation does not reflect a perfect inverse FS as in his diagrams; he appears to have accidentally skipped the interval 2 and extended the final interval to 21 in his notation.

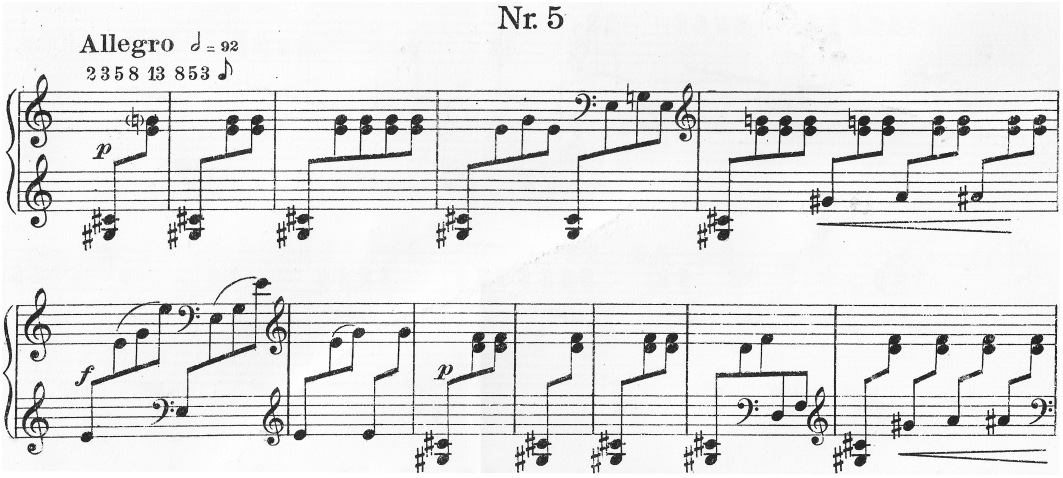
We can gain a clearer idea of why Schillinger was interested in FS and GR from the following quote from the mentioned book, referring to motifs based on summation sequences:

> These motifs have such a universal appeal that, whether they appear in folk music or in the work of an individual composer, they become universally accepted as definite crystallized symbols of musical expression.

In *The Mathematical Basis for the Arts* [10], a work whose "goal is to disclose the mechanism of creatorship as it manifests itself in nature and in the arts," he also gives examples of GFS in his chapter *Continuity*. In that book, however, he does not suggest an application of the sequences to any specific parameters, leaving them instead as raw compositional material.

3.3 Blacher's *Ornamente* (1950)

Boris Blacher's piano composition *Ornamente* [11], consists of seven studies in variable meters. In each study, the meter changes every measure according to a certain integer sequence. In the fifth study, dedicated to Rolf Liebermann, he utilizes FS. The piece consists of 113 measures. We observe Fibonacci metric summation and reversal without repetition in the sequence of meters, which consists of groups of $< 2, 3, 5, 8, 13, 8, 5, 3 >$ eighth notes repeated 14 times, followed by a final 2/8 meter that completes the symmetry. The piece begins as follows:



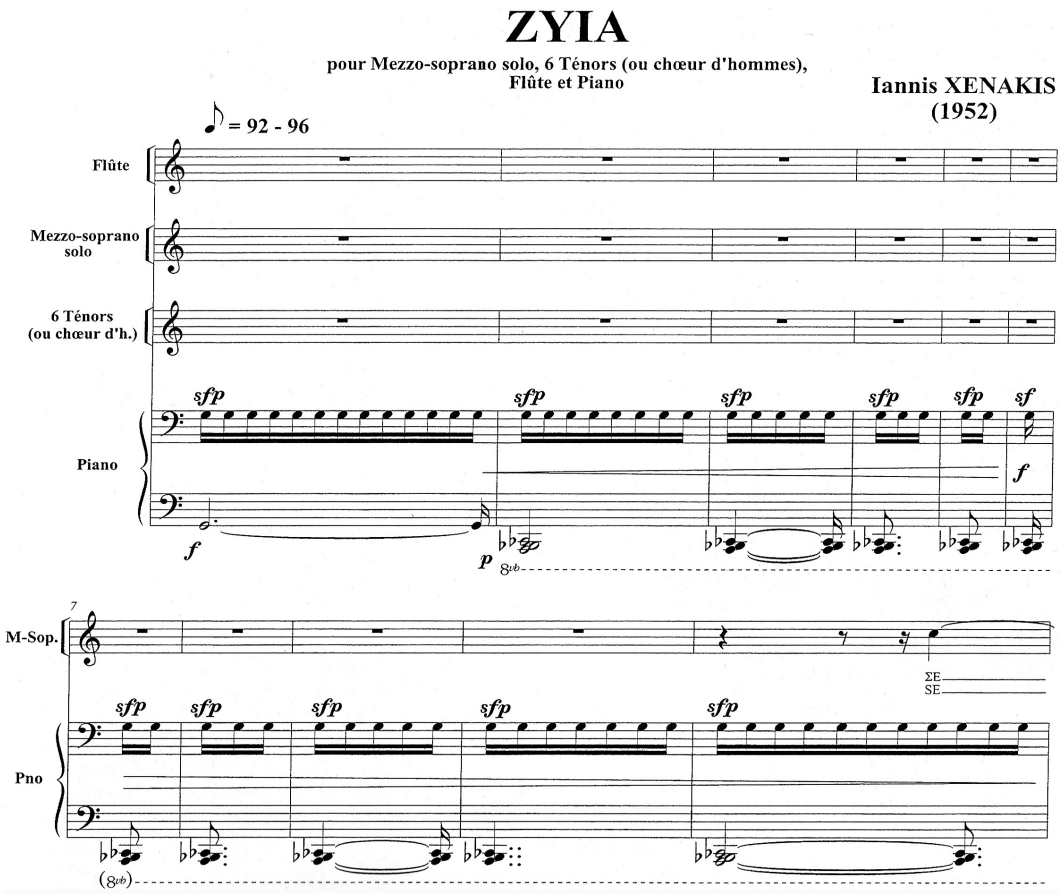
We gain some insight into why Blacher applied FS and other sequences in these studies from the following quote from the score:

> The idea of designing the metric process in such a way that every measure is governed by a different metric structure came from the realization that changes of meter often intensify the formal progression. If one bases the metric proportions on mathematical principles, namely on sequences or on combinatorics—in such a way metric progression is no longer a product of capriciousness or randomness. New symmetries on a higher level result, interesting overlaps of metric sequences with musical phrase, variable reprises and other things as well.

Blacher also used FS in his *Musik für Osaka* for the Expo '70 in Osaka, realized with the help of Rüdiger Rüfer. The piece utilized 33 loudspeakers and seven tracks of audio, with spatialization automated using an extra tape track. A description of the work and application of FS can be found online [12].

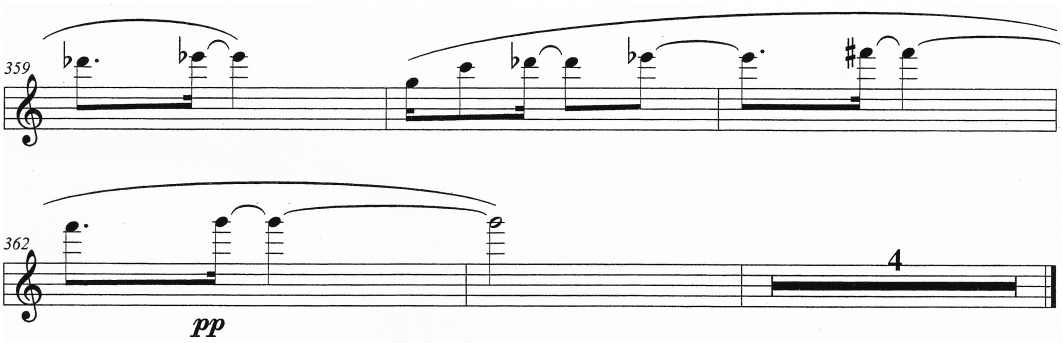
3.4 Xenakis' *Zyia* (1952)

Iannis Xenakis' first explicit application of a mathematical process to music was when he used FS in *Zyia* [13, 14]. The piece opens as follows [15]:



Here, we see an application of inverse Fibonacci metric summation in the pattern of meters in the first nine measures, which contain numbers of sixteenth notes derived from $< 13, 8, 5, 3, 2, 1, 2, 3, 5 >$. In this pattern, we also see the use of reversal without repetition. Xenakis breaks the pattern in the tenth measure with a meter containing seven sixteenth notes, shortly after which the soprano enters.

Further application of Fibonacci numbers to individual rhythmic values occurs throughout. At the end of the composition, in measures 360 to 363, the flute pattern utilizes Fibonacci pulse summation in the number of sixteenth notes $< 1, 2, 3, 5, 8, 13 >$ attributed to each note in the ascending phrase:



It has been speculated that Xenakis' interest in FS was sparked by Le Corbusier [16], with whom he was working at the time and who used them in his proportioning

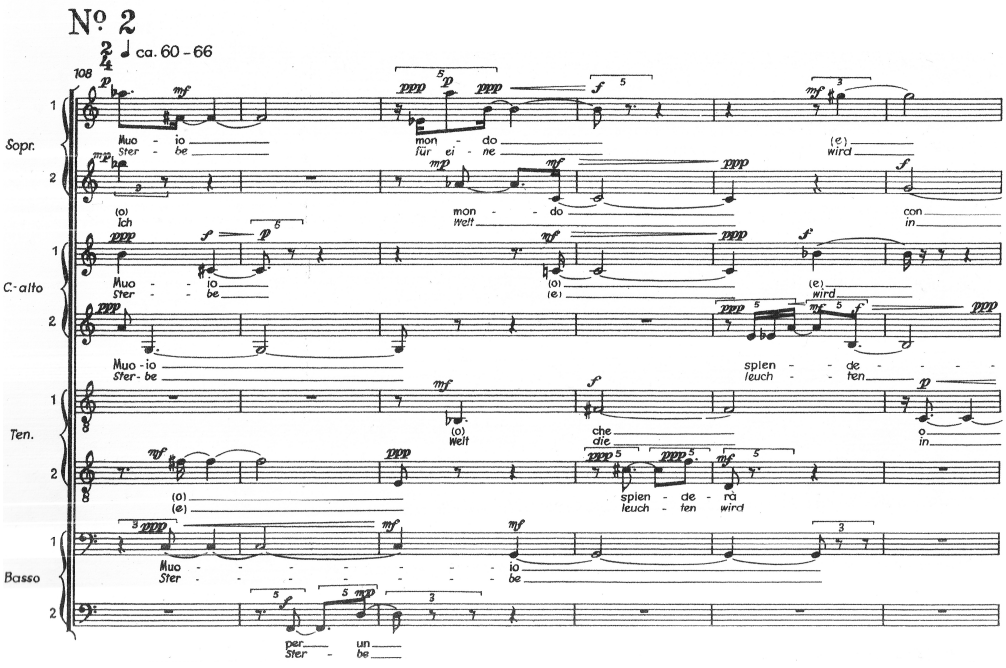
system the *Modulor*.

Xenakis discussed his use of FS and GR in *Anastenaria, Le Sacrifice* [3]. In the preface, he states:

> The golden section is one of the biological laws of growth. One finds it in the proportions of the human body; for example, the ratio of a person's height to the location of the solar plexus is the golden section … musical durations are created by muscular discharges that actuate the human limbs. It is clear that the movements of these limbs have a tendency to correspond temporally to the dimensions of these numbers. Thus the consequence: durations corresponding to the golden ratio are more natural for the movements of the human body … The golden number is irrational in mathematics. In music it is not possible (exception: electronic music). Providing durational values that a normal musician can count, I formed a [Fibonacci] sequence as just described.

3.5 Nono's *Il Canto Sospeso* (1956)

Luigi Nono utilized FS in the second movement of *Il Canto Sospeso* (for soloists, choir and orchestra) [17]. The text is taken from the farewell letters of resistance fighters condemned to death by the Nazis, and the work is dedicated "a tutti loro" [to all of them]. The first six measures are shown below:



In this movement, which Jonathan Kramer provides a detailed analysis of [18], continuous and unbroken four-part harmony occurs between the voices. Each of the four parts has its own base pulse value. The four pulse values used are: eighth note, triplet eighth note, sixteenth note, and quintuplet sixteenth. The row form used is the classic wedge: $< 0, 1, 11, 2, 10, 3, 9, 4, 8, 5, 7, 6 >$. There is always only a single row

occurring at any one point in time, and the notes of the row are always distributed through the four voices. The row always begins on A and is repeated 19 times. Kramer shows in a diagram how the Fibonacci rhythmic values are serialized and mapped to the pitches in measures 108 to 142, shifting the values of the sequence every time the row begins anew:

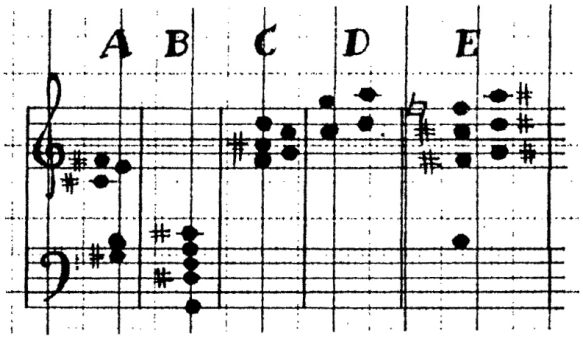
FIBONACCI DURATION SERIALIZATION IN Il Canto Sospeso, SECOND MOVEMENT

measure in which row form begins:	108	110	112	114	117	119	121	123	125	128	130	132	135	137	139
A	1	2	3	5	8	13	13	8	5	3	2	1	2	3	5
B♭	2	3	5	8	13	13	8	5	3	2	1	1	3	5	8
A♭	3	5	8	13	13	8	5	3	2	1	1	2	5	8	13
B	5	8	13	13	8	5	3	2	1	1	2	3	8	13	13
G	8	13	13	8	5	3	2	1	1	2	3	5	13	13	8
C	13	13	8	5	3	2	1	1	2	3	5	8	13	8	5
F♯	13	8	5	3	2	1	1	2	3	5	8	13	8	5	3
C♯	8	5	3	2	1	1	2	3	5	8	13	13	5	3	2
F	5	3	2	1	1	2	3	5	8	13	13	8	3	2	1
D	3	2	1	1	2	3	5	8	13	13	8	5	2	1	1
E	2	1	1	2	3	5	8	13	13	8	5	*3	1	1	2
E♭	1	1	2	3	5	8	13	13	8	5	3	*2	1	2	3

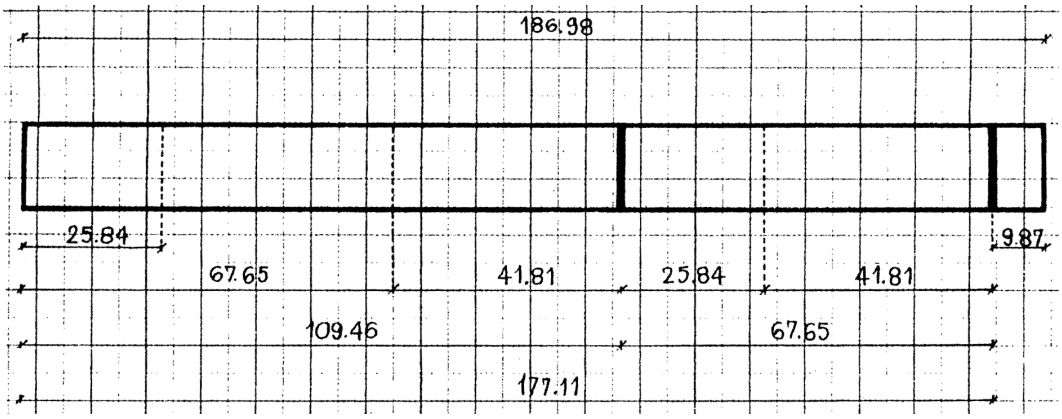
This is an unusually cryptic serial application of FS, but we can see the use of reversal with repetition in the sequence utilized $< 1, 2, 3, 5, 8, 13, 13, 8, 5, 3, 2, 1 >$.

3.6 Amenábar's *Los Peces* (1953–1957)

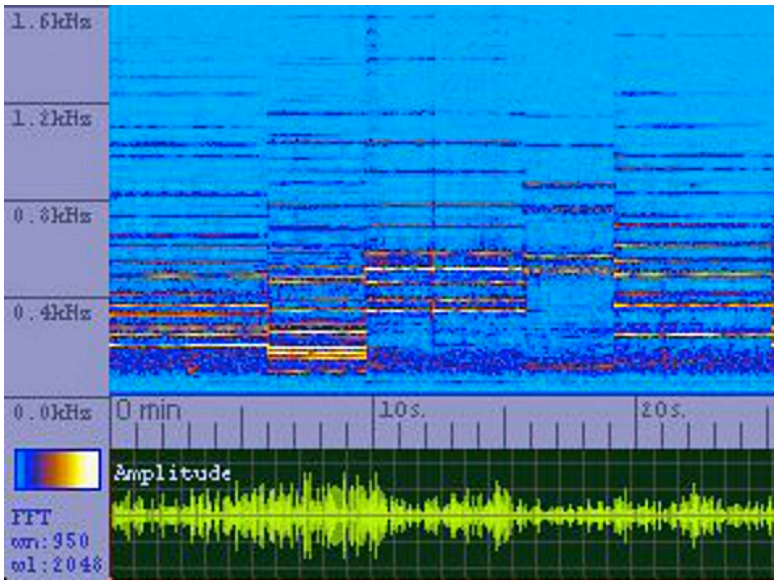
In collaboration with architect Pedro Buchard, Juan Amenábar applied FS to temporal proportion in a tape music piece entitled *Los Peces*, one of the earliest pieces of tape music in Latin America. According to the program notes, provided in an interview by Martín Alejandro Fumarola [19], the pitch material in the work consists of five chords, played on a piano, in which the attacks have been removed:



We note, however, that the second chord indicated in these program notes is incorrect: it sounds like the top two notes in the second chord are A-sharp and C, not A and C-sharp. Amenábar provides a glimpse of the metric structure in the program notes as well:



Each square on the graph paper used represents ten seconds. The durations of the sections are derived directly from Fibonacci numbers divided by 100. The first section indicated, for example, lasts 25.84 seconds, which is $F_{18}/100 = 2584/100$ seconds. A spectrogram of the first section of the work, available along with a sound file of the work online [20], is shown below:



The six chords heard in this excerpt, according to the letters given in the program notes, are <A, B, C, C, D, E>; their temporal proportions, or the Fibonacci numbers used, are $< 610, 377, 233, 377, 377, 610 >$, which add up to 2584. Heavy use of reversal with and without repetition is evident throughout and we see some unordered application

of FS as well.[3]

Through analog electronic means, Amenábar was able to approximate GR with great precision in this composition.

3.7 Krenek's *Fibonacci Mobile* (1964)

Despite all the author's attempts to get a hold of a score of Ernst Krenek's *Fibonacci Mobile* (for string quartet and piano four hands), he has remained unsuccessful. In an interview by Will Ogdon from 1972 [21], Krenek described his application of FS as follows:

> Another step forward in determinacy has to do with the concept of the mobile. In my orchestra piece *From Three Make Seven* I composed two blocks of music. Of each of these I made three versions at three different speeds, and each of these versions was orchestrated in three different ways (wind instruments, strings, and percussive devices), so that there were eighteen elements to work with, different but closely related to each other. In each element seven time points were indicated at which the elements could be synchronized. The plan of synchronization, the number and sequence of run-throughs and other aspects have to be decided upon before the performance—there is nothing left to improvisation. But the number of possible combinations is enormous.
>
> A similar idea is the basis of my *Fibonacci-Mobile*, a piece for string quartet and piano-four hands. Here the individual elements are proportioned according to a section of the Fibonacci series of numbers: 2 3 5 8 13 21 34, and again the shorter (faster) and longer (slower) elements may be combined in many different ways.

This description implies an unordered temporal application of FS and some aleatoric principles in the organization of the work.

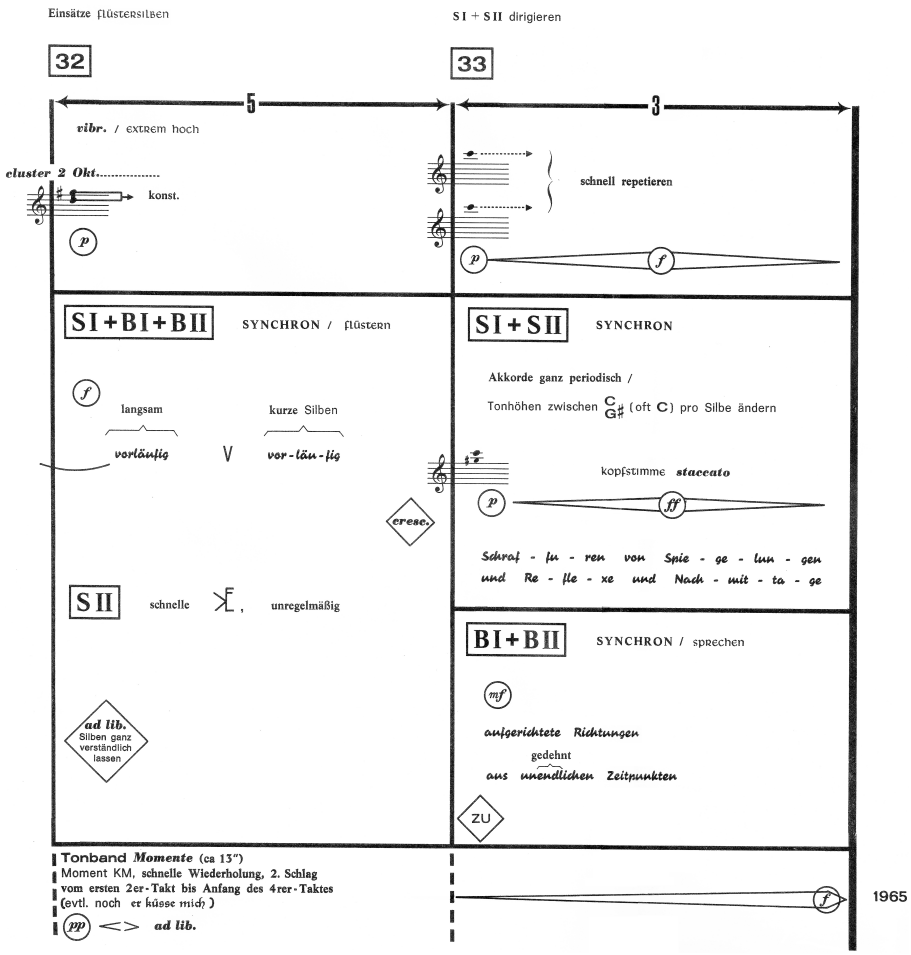
3.8 Stockhausen's *Mikrophonie II* (1965)

In 1965, Karlheinz Stockhausen finished *Mikrophonie II* (for choir, Hammond organ and four ring modulators), a work in moment form consisting of 33 moments [22]. In this work, an underlying pulse is chosen between 50 and 72 beats per minute that remains constant throughout the composition. FS is applied temporally—not to meter or rhythm, but directly to proportion; each moment lasts a Fibonacci number of pulses. The number of pulses in each of the 33 moments are as follows:

<5,3,144,34,89,8,8,34,34,21,13,13,8,8,21,55,8,55,21,55,13,13,21,13,13,8,8,34,89,8,21,5,3>

As can be seen, this is an unordered application of FS. The last two moments in the composition $< 5, 3 >$ are shown below:

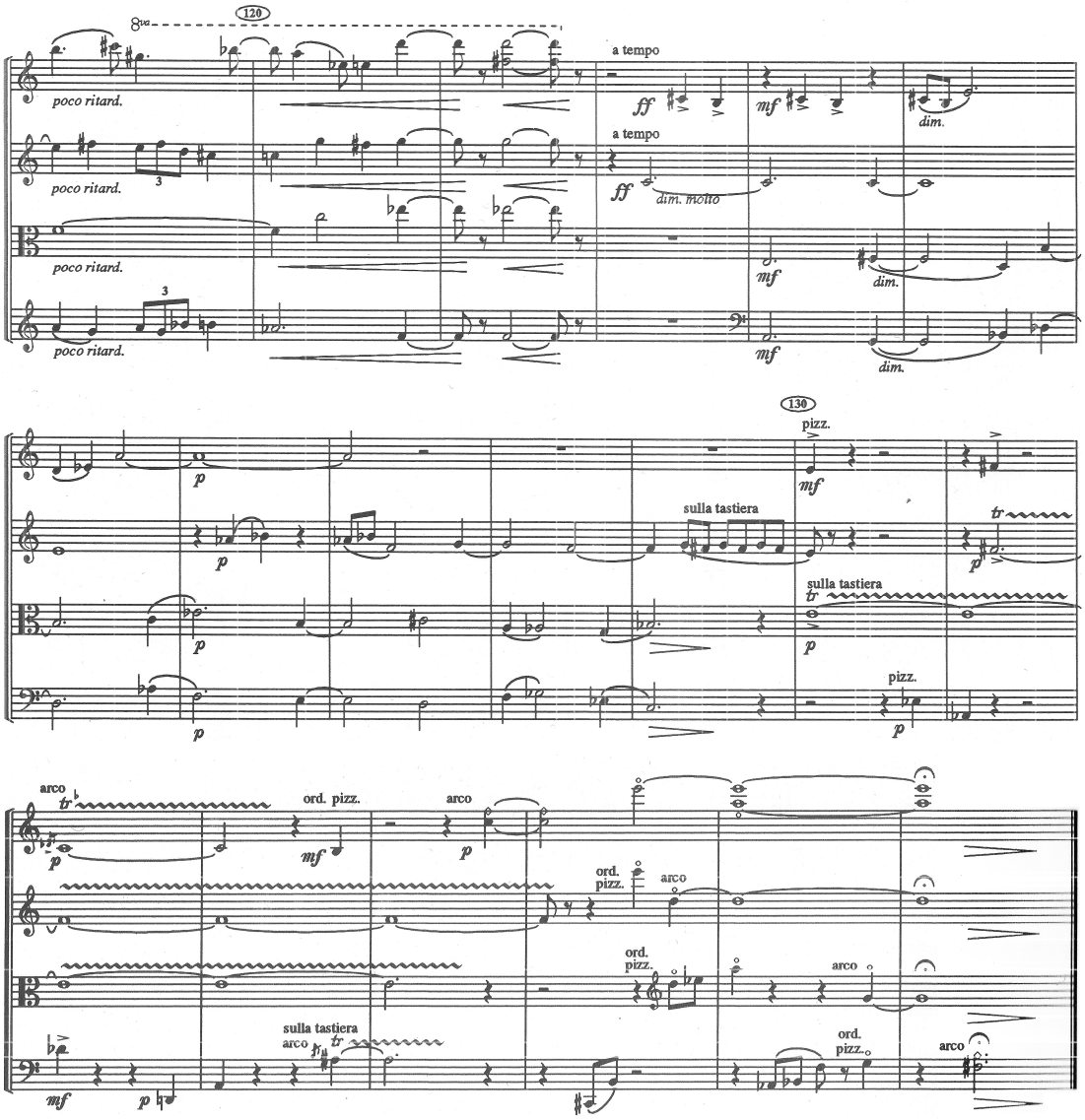
[3] An unordered application of FS can be regarded more as utilization of Fibonacci *numbers* rather than FS itself.



The text used in *Mikrophonie II* is Helmut Heißenbüttel's poem *Einfache grammatische Meditationen* (Simple Grammatical Meditations). Eight quotes of earlier Stockhausen works are heard in moments {2,5,8,11,17,19,20,32}—excerpts from *Gesang der Jünglinge*, *Carré*, and *Momente*. Stockhausen said in an interview with Piergiorgio Odifreddi in 2004 [23] that he used FS in many works, for the "duration of different sections as well as for the density of sound masses." In [18], Kramer discusses the temporal utilization of FS in *Klavierstück IX*, which was composed from 1954–1961.

3.9 Kokkonen's *Quartetto II* (1964–1966)

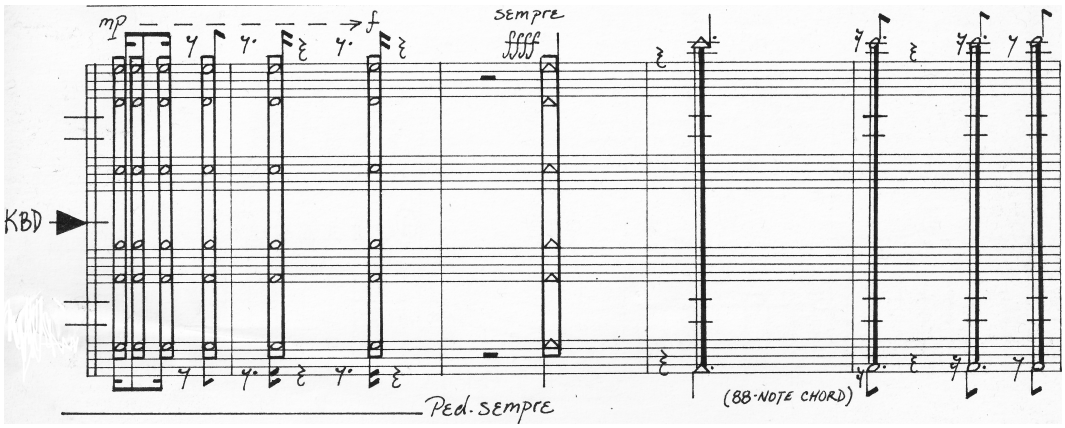
Joonas Kokkonen applied FS to proportion in the first movement in his second string quartet [24], a twelve-tone work. Edward Jurkowski provides a detailed analysis of the first movement, *Passacaglia*, in [25]. According to Jurkowski, the lengths of the sections defined by the theme of the passacaglia are derived from FS. In the third section in Jurkowski's analysis, for example, he points out how the length in measures of the sections defined by the theme of the passacaglia are $< 3, 5, 8, 13, 8, 5, 3 >$. This indicates the use of reversal without repetition; we also see some use of reversal with repetition in Jurkowski's analysis. While we do not have room to look at a large part of the score, we can at least see the tail end of this section $< 8, 5, 3 >$ in the final bars of the movement shown below, beginning at *a tempo*:



According to my own correspondence with the Finnish composer Mikko Heiniö, who was Kokkonen's student, "[Kokkonen] was very keen on the golden ratio."

3.10 Bunger's *Pianography, Fantasy on a Theme by Fibonacci* (1971)

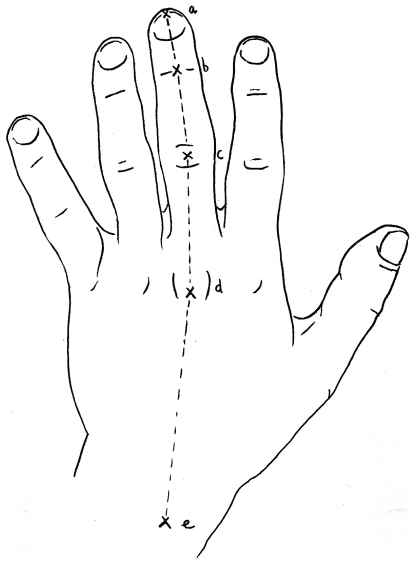
Richard Bunger made use of Fibonacci pulse summation in his work *Pianography, Fantasy on a Theme by Fibonacci* (for monaurally amplified piano with ring modulator and tape recorder) [26]. The composer states that ideally, the speakers of the amplification system surround the audience. The sound of the tape recorder, which is not connected to the amplification system, emanates directly from the piano. The ring modulator is utilized starting at 2 minutes and 7 seconds, set to a constant frequency of 330 Hz. Just before the entrance of the tape loop, near the end of the composition, the pianist plays clusters utilizing a specially constructed bar:



We observe the sequence $< 1, 2, 3, 5, 8, 13, 12, 13, 8, 5, 3, 2, 1 >$ in the number of sixteenth notes each attack is attributed (the last two attacks are not shown above). In using the number 12 in this sequence, which is not a Fibonacci number, perhaps the composer wanted to purposely break the pattern just before the 88-note chord occurs. Similar applications of FS occur throughout. An application of Fibonacci pulse summation utilizing reversal with and without repetition can be observed in measures 9 to 22.

3.11 Nørgård's *Canon for Organ* (1970–1972)

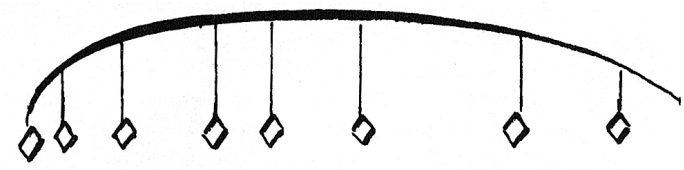
It was mentioned earlier that a direct application of GR is rare and—according to Xenakis—not possible in instrumental music. However, Per Nørgård applies GR directly to rhythmic values intended to be realized by a performer in his *Canon for Organ* [27], dedicated to Jesper Jørgen Jensen. In the preface to that work, he states that the "Golden Section is known as an essential, irrational division in the shapes of nature and art." Like Schillinger and Xenakis, he demonstrates an interest in the role of GR in biology, including a drawing in the score showing GR in the proportions of a human hand:



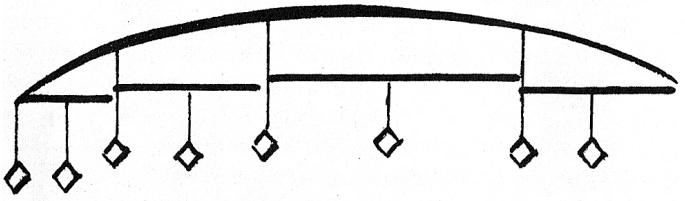
He summarizes the *Canon* as follows:

> 'Canon' presents for the first time The Golden Section transferred to the durations of tones and groups of tones. The Golden Section being an irrational number an adequate notation necessarily must be proportional. In order to make an organic adjustment to the demands of this notation possible, a metric version is published at the same time as the proportional versions. The values of the metric version are necessarily approximate (regarding all 'golden' durations), but the organist is recommended to study, practice, and possibly perform this version as a start. When he has arrived at an organic feeling of the rhythmical proportions, it will be the adequate time to start the practising directly from the proportional version.

Nørgård also gives examples of the types of rhythms that occur and how they are approximated with Fibonacci numbers in his "Commentary on The Proportional Notation of 'Canon'":



8 sustained tones in golden proportion (e.g.: 3:5:8:5:8:13:8:5) (stage ④ and ⑤).



4 pairs in golden proportion, each pair consisting of two tones of equal duration (e.g.: $2 \times 3/2 : 2 \times 5/2 : 2 \times 8/2 : 2 \times 5/2 = 1\frac{1}{2} : 1\frac{1}{2} : 2\frac{1}{2} : 2\frac{1}{2} : 4 : 4 : 2\frac{1}{2} : 2\frac{1}{2}$). (stage ⑥).

One can observe the use of reversal without repetition in the first example above.

The rhythmic and metric structure of the seven parts of *Canon*, each of which consists of eight lesser parts, is also derived from FS. We observe a proportional application of FS, and reversal with and without repetition in the seven parts:

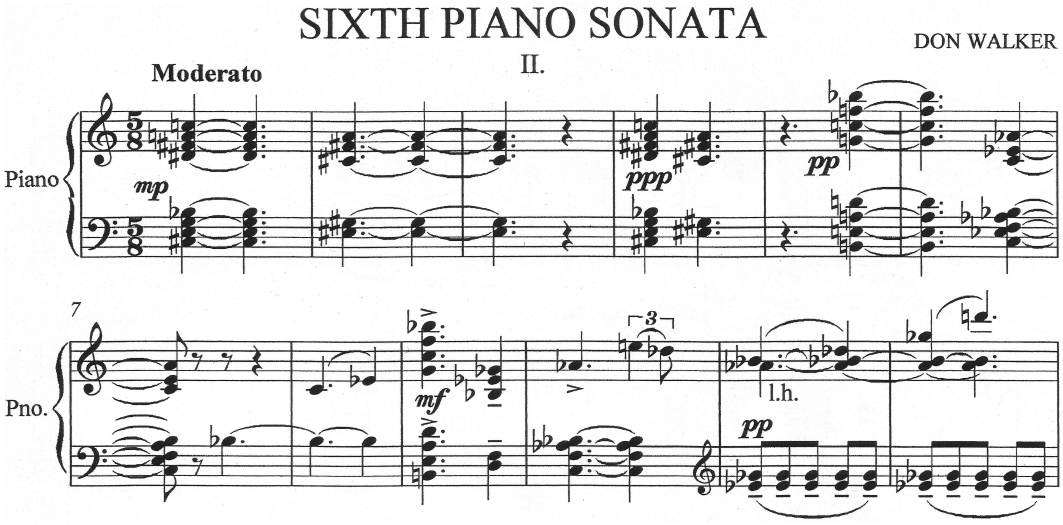
The mathematical relation on which the length of the stages is based in the various cycles is

1 : 1 : 1 : 1 : 1 : 1 : 1 : 1	in I,
1 : 2 / 2 : 1 / 1 : 2 / 2 : 1	in II,
2 : 3 : 5 : 3 / 3 : 5 : 3 : 2	in III,
3 : 5 : 8 : 5 : 8 : 13 : 8 : 5	in IV,
3 : 3 / 5 : 5 / 8 : 8 / 5 : 5	in V,
3 : 3 : 3 : 3 / 5 : 5 : 5 : 5	in VI, and
3 : 3 : 3 : 3 : 3 : 3 : 3 : 3	
(or: 1 : 1 : 1 : 1 : 1 : 1 : 1 : 1)	in VII.

The composer states, however, that various sections within this diagram are repeated and that section VII actually has 14 smaller parts.

3.12 Walker's *Sixth Piano Sonata* (1973)

Composer Donald Walker applied FS to pitch and rhythm in his *Sixth Piano Sonata* [28]. In the beginning of second movement, each note has a duration consisting of a Fibonacci number of eighth notes—with the exception of the triplet rhythm in measure ten:



The chords are also built using intervals that correspond to Fibonacci numbers. Counting the intervals of the first chord from the bottom up, for example, we obtain $< 3, 3, 3, 5, 3, 3, 3 >$. In measure ten, the chord in the left hand is made up of the intervals $<5, 3, 2>$, an inverse Fibonacci pitch sequence.

Walker has used FS and GR in various other compositions; his use of Fibonacci pitch sequences in *Business Music* was documented in the aforementioned article by Jonathan Kramer [18]. According to personal correspondence, Walker's use of FS in *Business Music* was for satirical purposes, because the composer found "Schillinger's descriptions of Fibonacci musical applications comical." However, after examining Bartók's use of FS, he began to see serious potential for the use of FS in his own work. He then composed his *Second String Quartet* (1972) and then his *Sixth Piano Sonata*. Walker states that he used FS and GR because they provided him with new approaches to the effort of creating new structures that were both novel and pleasing to his ear.

3.13 Chowning's *Stria* (1977)

John Chowning also applied GR directly in his computer-generated composition *Stria*. A thorough analysis of the composition was carried out by Matteo Meneghini [29]. Chowning, the inventor of FM synthesis, had been trying to find an inharmonic ratio to redefine the concept of the octave. In an interview with Curtis Roads [30], Chowning explains:

> I was looking for an inharmonic ratio such that the components would be powers of some basic ratio. It turns out that the Golden Mean (1.608) [sic] is such a number. If one has a [carrier to modulator] ratio that is 1 to

some power of the Golden Mean, then several of the low-order spectral side components are also powers of the Golden Mean.

What I did was draw an analogy between this inharmonic spectrum—including a frequency space where the pseudo-octave is at powers of the Golden Mean—and the harmonic series and tonality, where the low-order components of the harmonic series are also the principal intervals of the tonal system—the octave, the fifth, and so on. I drew this loose analogy and wrote some programs to help me compose, in particular to help me with the sound synthesis. It was not automatic composition by any means, but there were rules for determining the details of the structure, from the microsound level up to the level of a phrase.

In *Stria*, all frequency components are based on powers of the Golden Mean in the c:m ratios. Then I divided up the frequency space so there was some degree of complementarity.

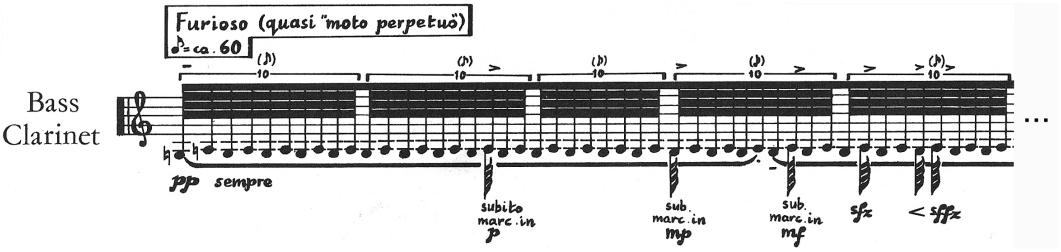
The pseudo-octave in *Stria* was divided into an equal temperament containing nine tones. A central frequency of 1000 Hz was utilized with three pseudo-octaves used above it, and five pseudo-octaves used below it. GR was applied to temporal parameters in *Stria* as well.

3.14 Ferneyhough's *Time and Motion Study I* (1977)

Brian Ferneyhough describes his *Time and Motion Study I* (for solo bass clarinet), dedicated to Harry Sparnaay, as follows [31]:

> *Time and Motion Study I* is built up around a core of two central principles: (1) the opposition and integration of two contrasted types of activity—the one rapid and regular, the other disjointed and gestural, with wide leaps and unpredictable rhythmic patterns; (2) the application, in various roles, of the proportions inherent in the Fibonacci Series (1, 1, 2, 3, 5, 8, 13, …). Right from the outset these worlds are brought into unmediated confrontation.

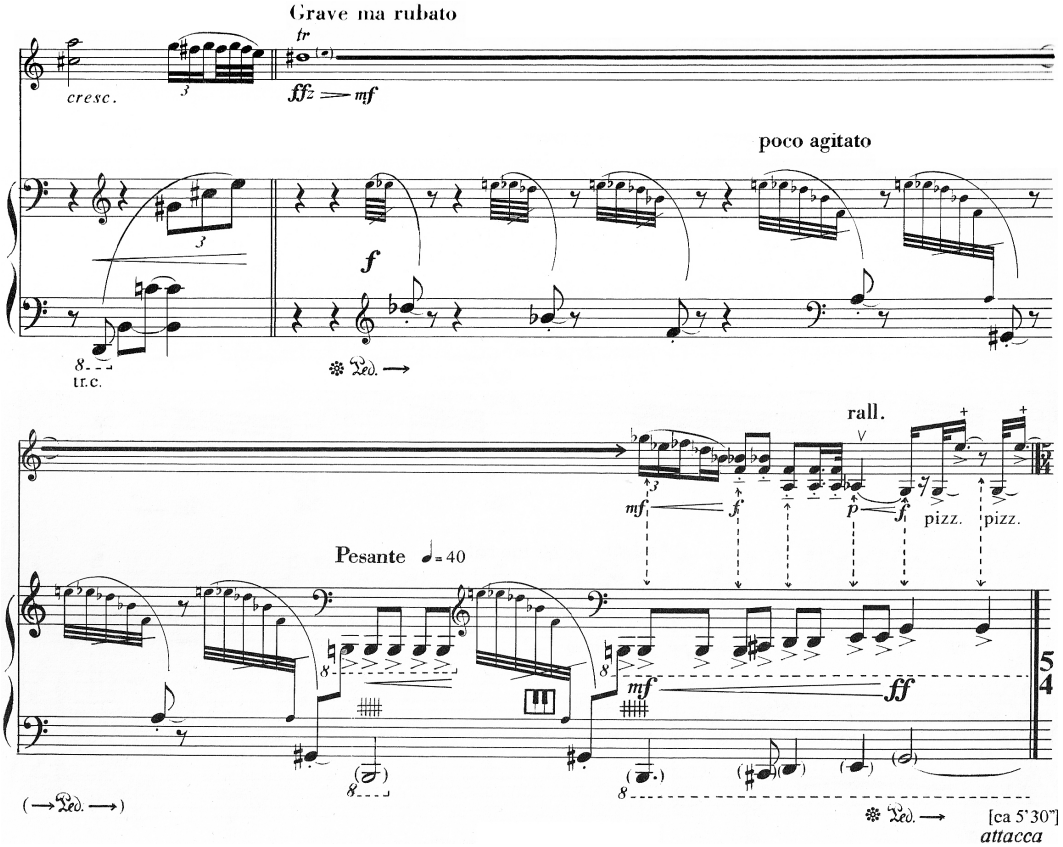
The presence of the inverse LS is clear in the pattern of accents and changing dynamics at the beginning of the work [32]. There is an accent on the 18th note played. Following this, there is an accent on the 11th note; this is followed by accents on the 7th, 4th notes, etc. This forms an inverse LS $< 18, 11, 7, 4, 3, 1 >$, as seen below:



After this, the pattern is broken and accents occur according to the pattern of the Lucas numbers minus one: $< 17, 10, 6, 3, 2 >$. Then he brings in the Lucas numbers $< 3, 4, 7, 11, 18 >$ again in the accent pattern, followed by the Lucas numbers plus one $< 19, 12, 8, 5, 4 >$ and then the Lucas numbers $< 1, 3, 4, 7 >$, after which the pattern breaks up at *subito pesante*.

3.15 Heiniö's *Duo per violino e pianoforte* (1979)

Interval patterns derived from FS similar to those described by Schillinger can be observed in Mikko Heiniö's *Duo per violino e pianoforte* [33]. In the piano part, we observe the following Fibonacci interval sequences in succession: $< 1, 2 >$, $< 1, 2, 3 >$, $< 1, 2, 3, 5 >$, $< 1, 2, 3, 5, 8 >$, $< 1, 2, 3, 5, 8, 13 >$, $< 1, 2, 3, 5, 8 >$, and finally $< 1, 2, 3, 5, 8, 13, 21 >$:

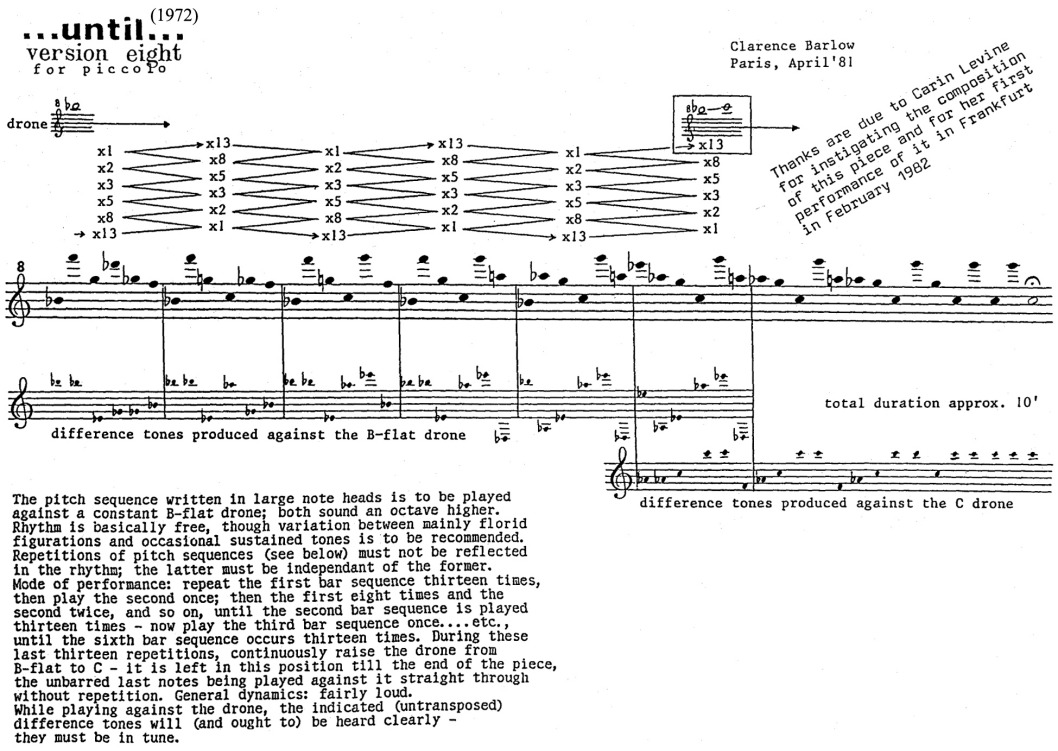


Unordered applications of FS to pitch occur throughout.

According to my own personal correspondence with the composer, he also used FS in his compositions *Tredicia* (1976) and *Notturno di fiordo* (1978). In *Tredicia*, the climax divides the piece into two parts with the ratio 8:5. Metric ratios such as 1:2, 2:3, 3:5, and so on occur throughout the work. The pitch system has nothing to do with FS or GR but is instead based on a twelve-tone chord. In *Notturno di fiordo*, the meter is also derived from FS and all harmonies and melodies are as well.

3.16 Barlow's *... until ... / version eight* (1981)

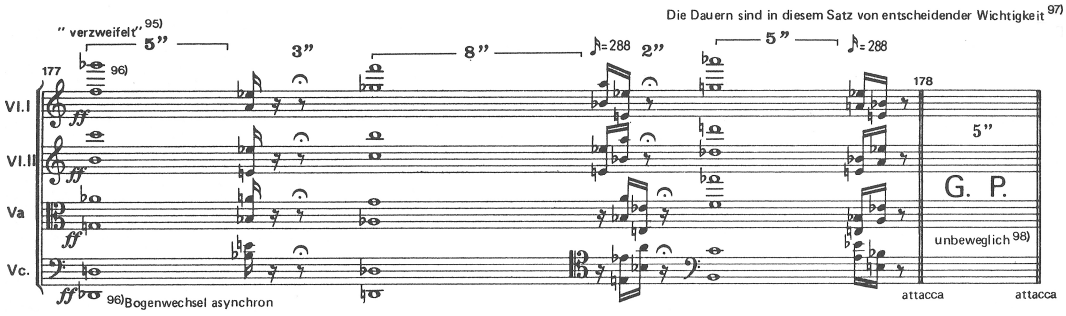
A unique application of FS occurs in Clarence Barlow's *... until ... / version eight* [34], for piccolo and drone. In this aleatoric work, the number of repetitions of a hexachord are determined by FS. The composer has granted me permission to print the score in its entirety here:



According to personal correspondence, Barlow's interest in FS in this piece has to do with the idea of what he describes as a "quasi-exponential fade" according to FS proportions between the alternating measures, as indicated by the zigzagging arrows in the score; e.g. the first measure slowly "fades" into the second measure according to FS. For Barlow, FS is "a technical tool serving an aesthetic purpose." Barlow also used FS in his *Tschelpowisch* (1968–1971) [35].

3.17 Gielen's *Streichquartett* (1983)

In the third movement of his *Streichquartett* [36], Michael Gielen applies FS to temporal proportions. Similar to Stockhausen's application in *Mikrophonie II*, the numbers of FS are not used in order. We can see the use of $< 5, 3, 8, 2, 5, 2, 5 >$ in the first measures of the movement:

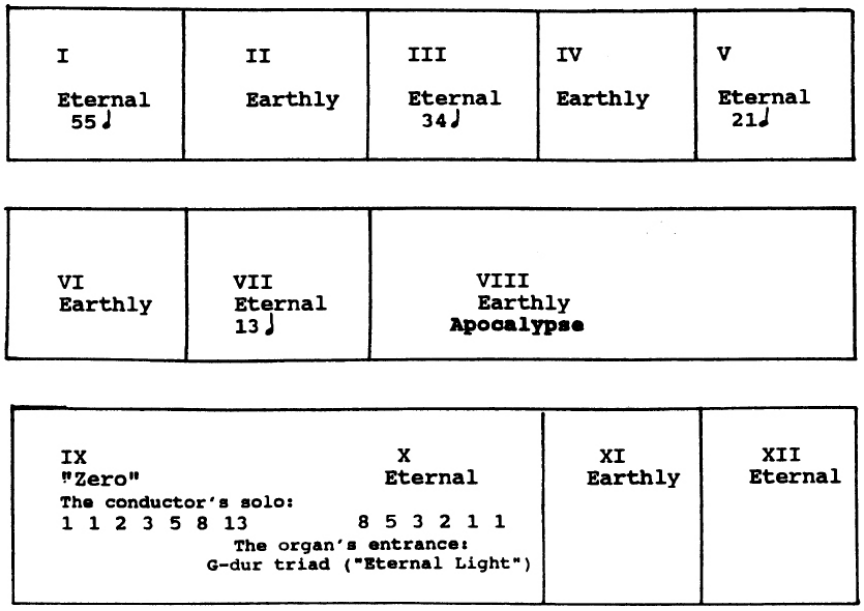


The instructions in German at the top right translate to "The durations are of crucial importance in this movement." Each of the six grand pauses in the work last a Fibonacci

number of seconds $< 5, 8, 13, 8, 5, 8 >$. Some proportions utilized in this movement are, however, clearly not Fibonacci-related (e.g. $< 4, 9, 5, 3 >$ in measure 188).

3.18 Gubaidulina's *Stimmen ... Verstummen ...* (1986)

In 1986, Sofia Gubaidulina completed a symphony in 12 movements titled *Stimmen ... Verstummen ...* [37]. According to the composer, all odd movements are organized according to FS, while the even movements are freely composed. Like much of Gubaidulina's work, this composition contains symbolism. Those movements organized using FS are supposed to represent a "harmonic, eternal world," whereas the other movements symbolize a "disharmonic, earthly world." The following graphic, included with an interview with the composer by Vera Lukomsky [38], depicts the twelve movements of the piece:

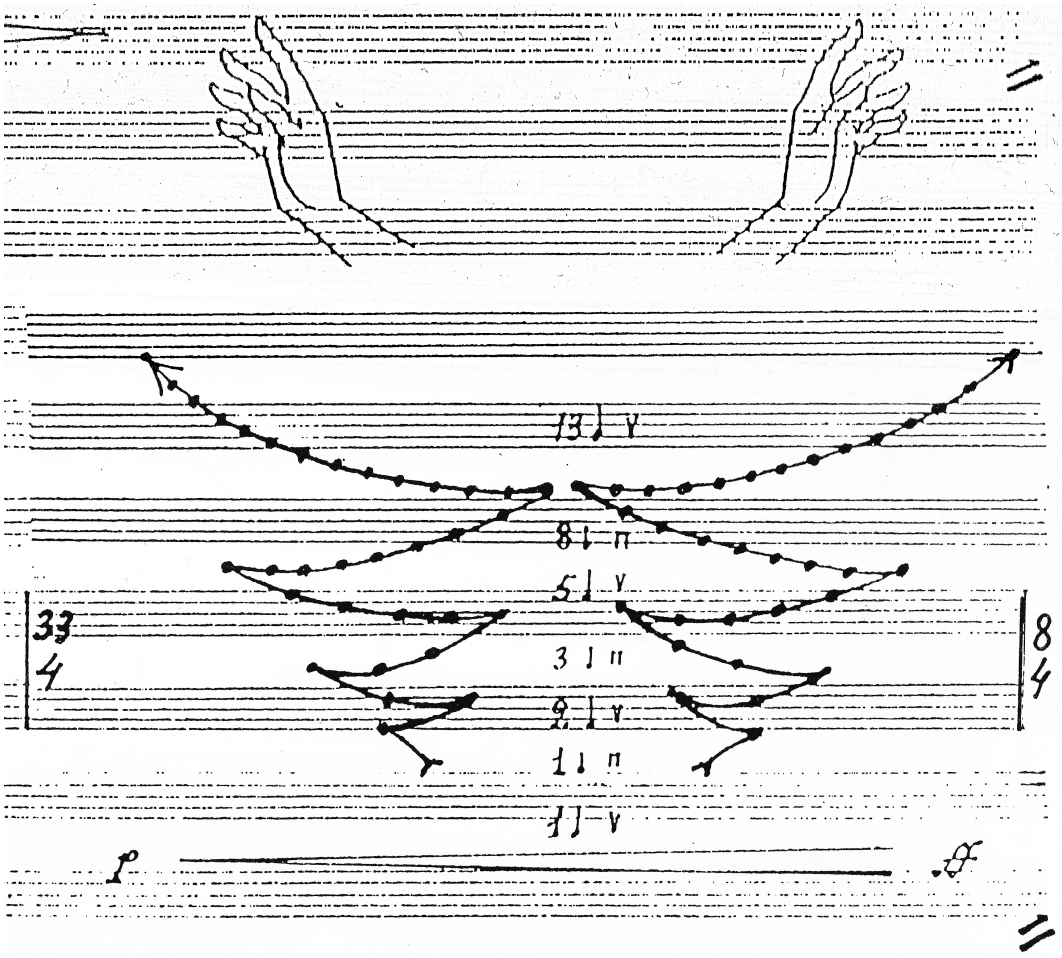


The ninth movement contains a very unique application of FS: a solo for conductor (with occasional percussion or organ accompaniment) that is mostly silent in which the conductor's gestures "erect rhythmic edifices." The composer describes the ninth and tenth movements, shown to be interconnected in the previous graphic, as follows:

> The Ninth [sic] movement is a 'rest': it is a solo for the conductor. It is as if the music had come to 'zero': in the first movement there was [sic] 55 quarters, in the third—34, in the fifth—21. [sic] in the seventh—13, and, finally, in the ninth—zero.
>
> The 'zero' is represented by the conductor's organizing time according to the Fibonacci series: (1, 1, 2, 3, 5, 8, 13). Then he 'holds' 13; at this time the organ enters with a major triad in the high register. For a while, the conductor continues to 'hold' that pyramid of Fibonacci numbers and then starts moving down the Fibonacci series, from 13 to 1.

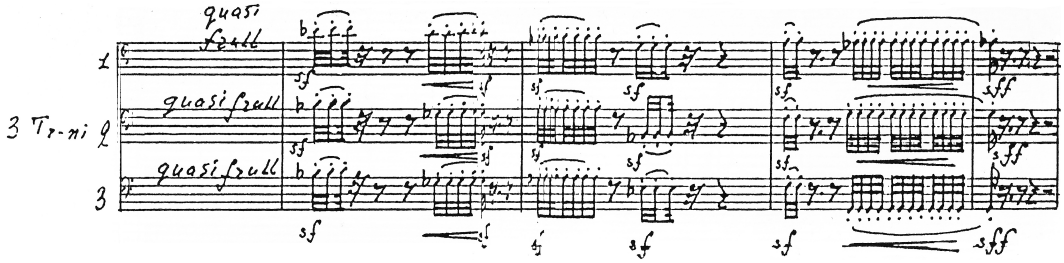
This indicates the use of reversal without repetition.

An example of one measure from the ninth movement is shown in the following graphic [37]:



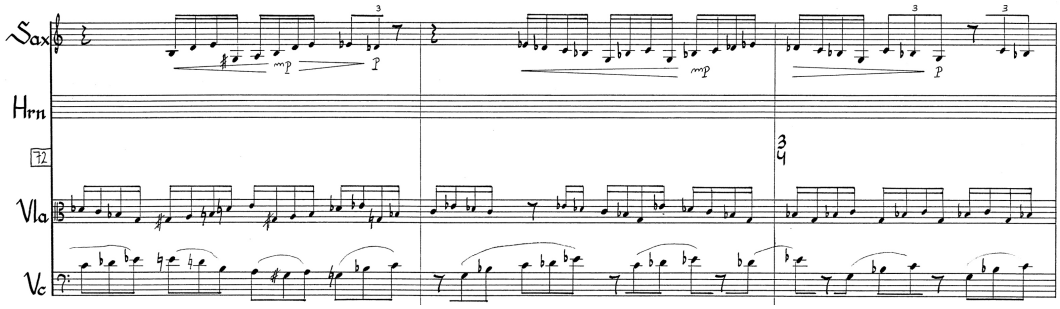
This gestural application of FS with pulse summation is unique among the pieces analyzed in this section.

In the last movement of the work, FS is applied to the number of attacks in certain gestures—the numbers are nonetheless utilized out of order for the most part. This can be seen in the timpani parts on p. 192 of the score:



3.19 Tómasson's *Octette* (1987)

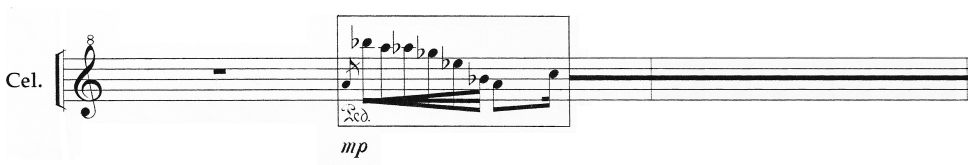
According to personal correspondence, Haukur Tómasson applied FS and GR to multiple musical parameters in *Octette* (for flute, clarinet, alto saxophone, French horn, viola, violoncello, piano and percussion) [39]. Almost all of the intervals that occur in the piece are derived from FS. Their arrangement seems to be intuitively defined, however, although we see some isolated occurrences of reversal without repetition (e.g. measure 76 in the piano). The use of intervals based on Fibonacci numbers can be seen in measures 72–74, where the only intervals that occur are {1, 2, 3, 5, 8}:



According to the composer, in addition to applying FS to pitch in this work, he also applied FS to various other parameters. The number of players is, for example, a Fibonacci number. FS was also applied to note durations (which seems less apparent), to the lengths of sections, and to meter and the number of iterations of notes (as can be seen at the end). Tómasson thinks of FS as a tool that helps him in compositional organization. In addition, he likes the rhythms and intervals that FS can generate because they approximate GR. He also used FS in his *Eco del passato* (for flute and harpsichord), *Spiral* (for 13 players) and *Annual Ring* (for ensemble).

3.20 Blumenfeld's *Diluvial* from *Illuminations: Symphonic Fragments After Rimbaud* (1992)

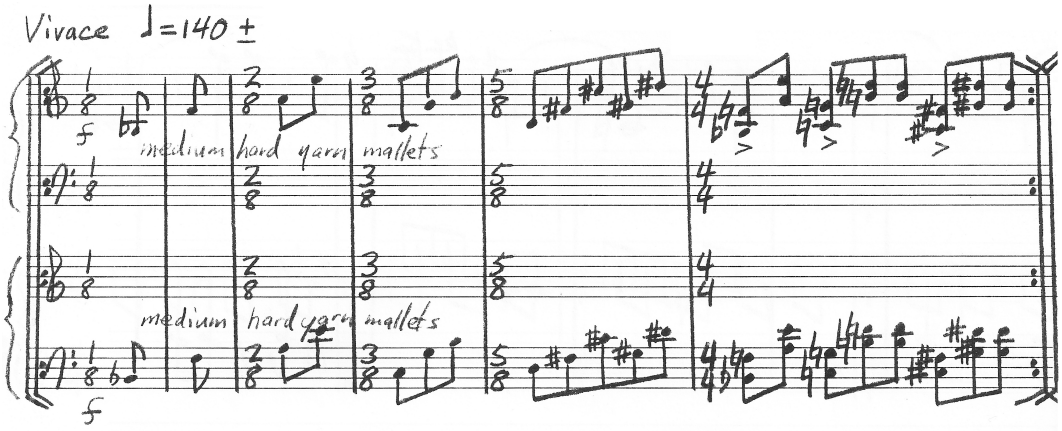
In the score for Harold Blumenfeld's orchestral composition *Diluvial* [40], based on poems by Arthur Rimbaud published in 1886 and dedicated to Gerhard Samuel, one finds the words "Composed after the Fibonacci Series" below the dedication. No description is given of how FS was applied. The metric and proportional structure of the composition does not come across as particularly rigorous (due to the use of accelerandi, tempo ranges, etc.). One can, however, observe the use of a Fibonacci pitch sequence starting in the 6th measure of section **P**, where the celesta plays $< 1, 2, 3, 5, 8, 13 >$ embedded in a descending repeated pattern. The pattern is interrupted, however, by the first note and the last two notes (the grace note A and the last A and C in the pattern), which are not part of the sequence:



In the two measures before section **O**, we see the descending Fibonacci pitch sequence $< 0, 1, 1, 2, 3 >$ in the horns as well. Most other pitch material in the composition does not appear to be related to FS.

3.21 Hill's *Abstract Relativity* (1999)

In a percussion duet for two marimbas and a vibraphone by William Hill, Fibonacci metric summation is utilized in multiple sections. We can see the use of $< 1, 1, 2, 3, 5, 8 >$ in the marimba parts in the opening bars of the piece:

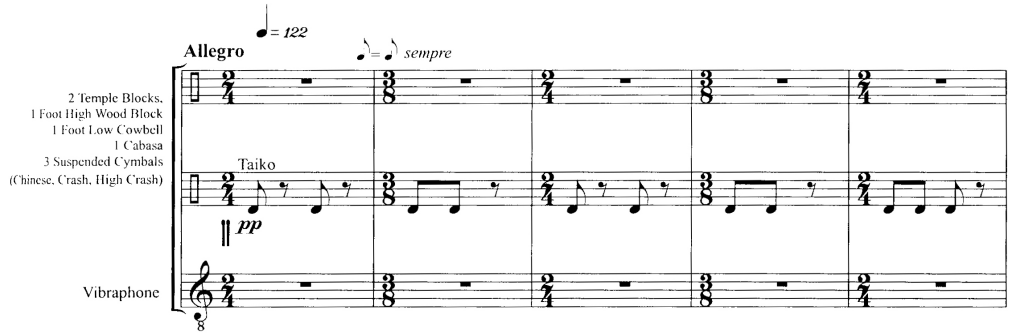


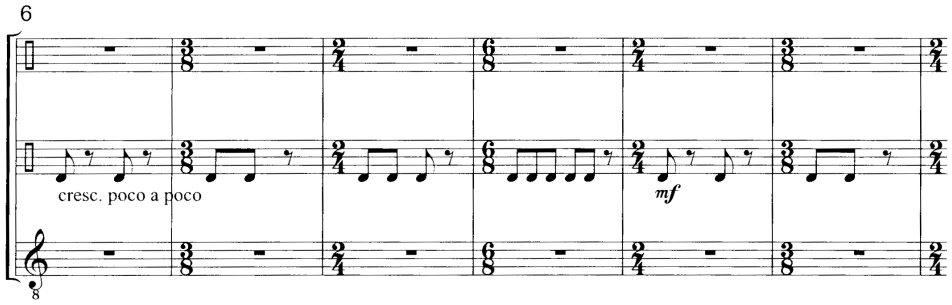
Similar application of FS to meter occurs throughout. We also see the utilization of reversal with repetition in the metric application of FS in the last 16 measures, where the number of eighth notes in each measure follows the sequence $< 1, 1, 2, 3, 5, 5, 3, 2, 1, 1, 1, 1, 2, 3, 5, 1 >$.

According to the introduction in the score [41], this composition is inspired by the work of Leonardo Fibonacci, Hieronymus Bosch, and Arnold Schoenberg. It integrates twelve-tone rows and juxtaposes "contorted dissonance and lyric consonance much as Bosch placed his visions of heaven and hell side by side."

3.22 García i Soler's *Scherzo con Fibonacci* (2002)

Ramón García i Soler uses FS in the rhythmic pattern in the opening measures of his *Scherzo con Fibonacci* for solo percussion [42]. The first section of the work, the beginning of which is shown below, contains the rhythmic patterns $< 1, 1, 2 >$, $< 1, 1, 2, 3 >$, $< 1, 1, 2, 3, 5 >$, and $< 1, 1, 2, 3, 5, 8 >$. The numbers in the sequences are punctuated by eighth note rests:

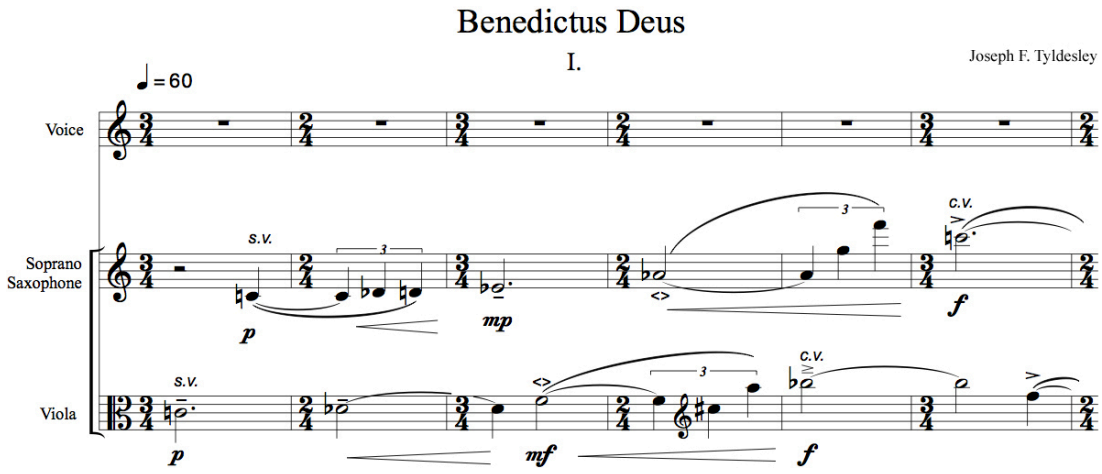


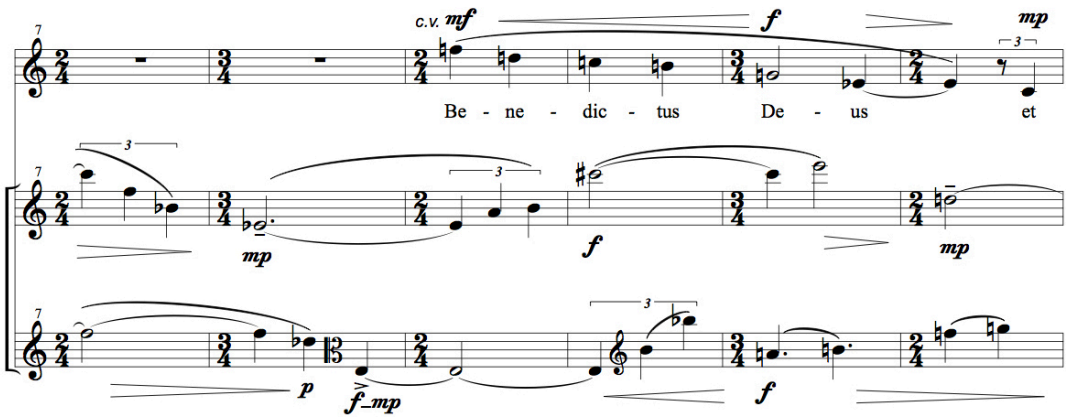


The work subsequently makes use of many other rhythmic patterns that do not appear to be derived from FS.

3.23 Tyldesley's *Benedictus Deus* (2005)

Joseph F. Tyldesley's *Benedictus Deus* (for soprano, soprano saxophone and viola) is the only sacred work analyzed in this survey [43]. The work consists of eight movements, each of which is a setting of a verse of Ephesians (1:3–10). Each movement contains 144 notes, utilizing a complete Fibonacci system modulo 12 (each of the ten sequences is used once), as discussed in section 2.1 of this article. The Fibonacci pitch class sequences are distributed among the three parts throughout (there is never more than a single pitch class sequence being utilized at once). In the first movement, C = 0, and the piece begins with <0>—the only Fibonacci pitch class sequence with a period of 1 (shown in the top row of the table in 2.1). Tyldesley then utilizes the Fibonacci sequence modulo 12 $< 0, 1, 1, 2, 3, 5, 8, 1, 9, 10, 7, 5, 0, 5, \cdots >$, as can be seen in the score below (the score is in C):

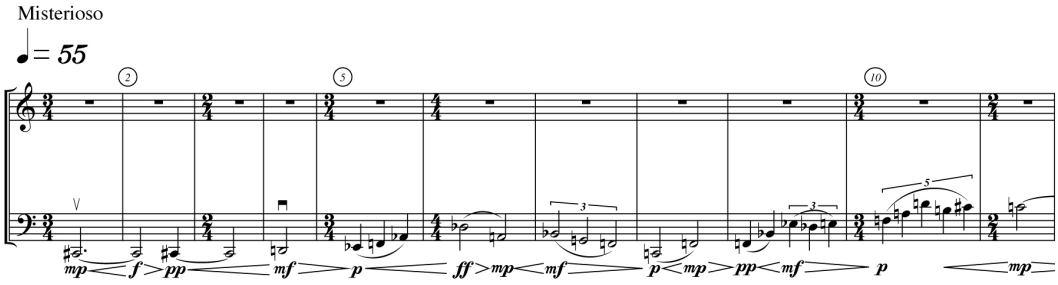




According to personal correspondence, the metric scheme was created through an additive, recursive process. The groupings of meters in this movement are indicated here through wedge brackets (the integers indicate the number of quarter notes in the meter): <3>, <2>, <3,2>, <2,3,2>, <3,2,2,3,2>, <2,3,2,3,2,2,3,2>, <3,2,2,3,2,2,3,2,3,2,2,3,2>, <2,3,2>. Notice that, with the exception of the last sequence in wedges, each group is the concatenation of the previous two groups, and that each group therefore contains a Fibonacci number of meters. This is similar to Fibonacci metric summation.

3.24 Odai's *Passacaglia di Fibonacci* (2008–2010)

Kiyomitsu Odai's orchestral work *Passacaglia di Fibonacci* is the most multifaceted application of FS we will encounter in this survey [44]. In the opening bars of a canon reminiscent of the first movement of Bartók's *Music for String Instruments, Percussion and Celesta*, the theme consists of one period of an ascending Fibonacci pitch class sequence where the note C = 0, beginning $< 1, 1, 2, 3, 5, 8, 1, 9, 10, 7, 5, 0, 5, \cdots >$:



There are 610 quarter notes in the work, and strict tempo is kept throughout. Major sections consist of Fibonacci numbers of quarter notes. A climax is reached on the 378th quarter, approximating GR with great precision (measure 124). Meters containing exclusively Fibonacci numbers of quarter notes occur throughout (instead of using an 8/4 meter the composer uses two 4/4 meters). In the metric structure, we see heavy use of reversal with and without repetition, and some unordered application of FS as well. Starting in measure 135, the orchestra "pronounces" the syllables "Fi," "bo," "na," "cci" through spectral representation: the consonances are produced by short grace notes on the beat, and the vowels by longer notes.

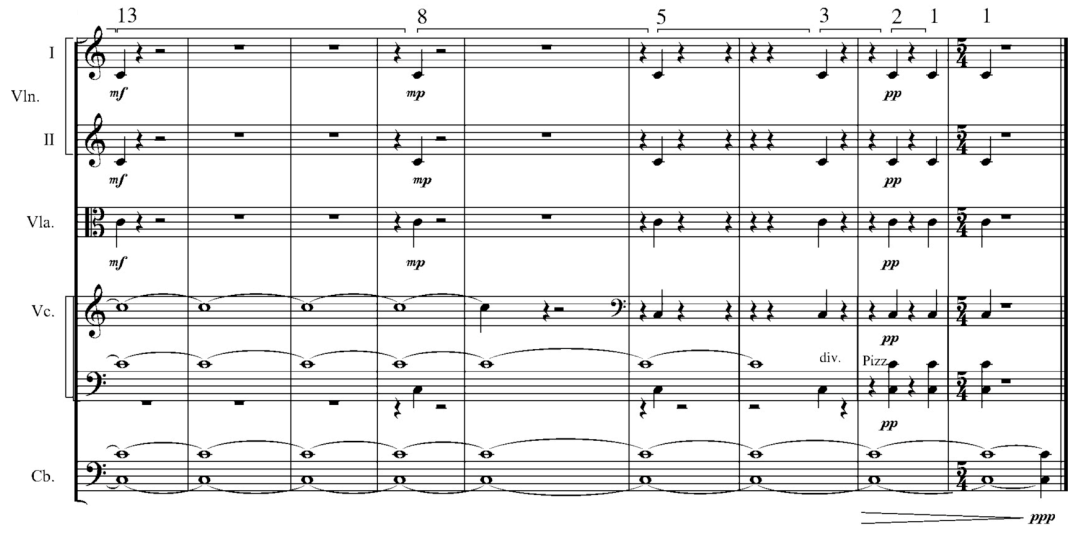
According to personal correspondence, Odai compares FS to a landscape. He has used FS and GR in nearly a dozen works—though recently less frequently—and is fascinated by the principle of infinite convergence and "the balance between symmetry and asymmetry."

3.25 Choi's *Dream of a Thousand Keys: A Concerto for Piano and Orchestra* (2011)

Da Jeong Choi describes the application of FS in her piano concerto *Dream of a Thousand Keys* in her dissertation [45] as follows:

> The Fibonacci proportions not only determine the form of the entire work but also the structure, tempo, meter, duration, rhythmic patterns, phrase length, and pitch collections.

In the second movement of this work, dedicated to her father Nak Joong, we observe the use of inverse Fibonacci pulse summation in the final measures 268 to 276:



The pitch material in the work is based on a twelve-tone row derived from the Fibonacci modulo 12 pitch class sequence. The pitches of the twelve-tone row used are shown below in bold in the order in which they occur, showing their derivation from a Fibonacci pitch class sequence—the final pitch class **6**, which never actually occurs in the Fibonacci sequence modulo 12, is appended to the end to complete the row:

0 1 1 **2 3 5 8** 1 **9 10 7** 5 0 5 5 10 3 1 **4** 5 9 2 **11** 1

According to personal correspondence, Choi takes great interest in mathematics and considers it to be one of the greatest tools in constructing her compositions. There is an orchestral version of the work entitled *From Dawn to Sunset* in which the same application of FS occurs. In addition to the use of FS, the work also integrates elements of traditional Korean music, jazz and aleatorics. This is the only work as of yet in which Choi has utilized FS.

4 Trends in the Utilization of FS and GR

In order to gain some oversight of the multitude of various applications we have seen in these analyses, we will construct a table that indicates the types of applications we have observed. Only those applications that we have noted in the pieces analyzed will be indicated in the table.

Because the applications we have seen are so diverse and numerous, it will be necessary to group certain applications together. Use of the inverse GFS and GFS are not differentiated in this table. Although metric and rhythmic applications can also be considered smaller scale proportional applications, the category "Proportional Application of GFS" in the table refers to larger scale temporal structure, as described earlier in 2.4. Unordered pitch and temporal/proportional applications of GFS (where the numbers of GFS appear out of order) are indicated in the categories "Other Pitch Application of GFS" and "Other Temporal Application of GFS." Only those applications in which at least three numbers of a GFS $< G_n, G_{n+1}, G_{n+2} >$ are utilized in order are considered to be ordered applications, unless such an ordering appears to be coincidental and isolated.

As can be seen in the table, temporal applications of GFS are most common, and have taken a variety of contrasting forms. The application of GFS to pitch and the direct application of GR, on the other hand, are relatively uncommon. Pulse summation appears to be more common than attack summation; Bartók was the only composer observed to have use attack summation, and the rhythmic irregularity in his specific application was discussed earlier. We only saw a single, cryptic, serial application of FS. Reversal was utilized in nearly half of the compositions analyzed, and reversal without repetition appears to be more common than with. We observed no applications of GFS or GR to dynamics or spatial location.

composer (year)	Pitch Sequences Based on GFS	Pitch Class Sequences Based on GFS	Other Pitch Application of GFS	Attack Summation Based on GFS	Pulse Summation Based on GFS	Metric Summation Based on GFS	Proportional Application of GFS	Other Temporal Application of GFS	Serial Application of GFS	Reversal With Repetition	Reversal Without Repetition	Direct Application of GR
Bartók (1936)				×							×	
Schillinger (1941)	×	×										
Blacher (1950)						×					×	
Xenakis (1952)					×	×		×			×	
Nono (1956)									×	×		
Amenábar (1957)							×	×		×	×	
Krenek (1964)								×				
Stockhausen (1965)								×				
Kokkonen (1966)							×			×	×	
Bunger (1971)					×					×	×	
Nørgård (1972)							×	×		×	×	×
Walker (1973)	×		×					×				
Chowning (1977)												×
Ferneyhough (1977)								×				
Heiniö (1979)	×		×									
Barlow (1981)								×				
Gielen (1983)								×				
Gubaidulina (1986)					×			×			×	
Tómasson (1987)			×					×			×	
Blumenfeld (1992)	×											
Hill (1999)						×				×		
García i Soler (2002)								×				
Tyldesley (2005)	×	×						×				
Odai (2010)	×	×				×	×	×		×	×	
Choi (2011)			×		×							
total occurrences	6	3	4	1	4	4	4	14	1	7	10	2

References

[1] Steven Vadja. *Fibonacci and Lucas Numbers, and the Golden Section*. Mineola: Dover, 2008.

[2] Parmanand Singh. The So-Called Fibonacci Numbers in Ancient and Medieval India. *Historia Mathematica*, 12:229–244, 1985.

[3] Iannis Xenakis. Preface to *Anastenaria*. Paris: Boosey & Hawkes, 1954.

[4] Stefan Andrus Burr. On Moduli for Which the Fibonacci Sequence Contains a Complete System of Residues. *The Fibonacci Quarterly*, 9(5):497–504, 1971.

[5] Béla Bartók. *Music for String Instruments, Percussion and Celesta*. New York: Boosey & Hawkes, 1937.

[6] Roy Howat. Lendvai and the Principles of Proportional Analysis. *Music Analysis*, 2(1):69–95, March 1983.

[7] Tibor Bachmann and Peter J. Bachmann. An Analysis of Béla Bartók. *Musical Quarterly*, 65:72–82, 1979.

[8] Edward A. Lowman. Some Striking Proportions in the Music of Béla Bartók. *The Fibonacci Quarterly*, 9:527–537, 1971.

[9] Joseph Schillinger. *The Schillinger System of Musical Composition*. New York: Carl Fischer, 1946.

[10] Joseph Schillinger. *The Mathematical Basis of the Arts*. New York: Philosophical Library, 1948.

[11] Boris Blacher. *Ornamente*. Berlin: Ed. Bote & G. Block, 1951.

[12] Technische Universität Berlin. Musik für Osaka: Gestaltung des Kugelraums. http://www2.ak.tu-berlin.de/Geschichte/themen/Osaka.html, 2013.

[13] James Harley. *Xenakis: His Life in Music*. London: Routledge, 2004.

[14] Makis Solomos. Du projet bartokien aus son. L'évolution du jeune Xenakis. In *Présences de Iannis Xenakis*, pages 15–29. Paris: Centre de documentation de la musique contemporaine, 2001.

[15] Iannis Xenakis. *Zyia*. Paris: Editions Salabert, 2013.

[16] Agostino Di Scipio. Formalization and Intuition in Analogique A et B. In *Definitive Proceedings of the "International Symposium Iannis Xenakis"*, Athens, May 2005.

[17] Luigi Nono. *Il Canto Sospeso*. Mainz: Ars Viva Verlag, 1957.

[18] Jonathon Kramer. The Fibonacci Series in Twentieth-Century Music. *Journal of Music Theory*, 17:110–148, 1973.

[19] Martín Alejandro Fumarola and Juan Amenábar. Electroacoustic Music Practice in Latin America: An Interview with Juan Amenábar. *Computer Music Journal*, 23(1):41–48, 1999.

[20] La fondation Daniel Langlois. Juan Amenábar Ruiz: *Los Peces*, 1957. http://www.fondation-langlois.org/html/e/page.php?NumPage=1594, 2012.

[21] Will Ogdon and Ernst Krenek. Conversation with Ernst Krenek. *Perspectives of New Music*, 10(2):102–110, 1972.

[22] Karlheinz Stockhausen. *Mikrophonie II*. London: Universal Edition, 1974.

[23] Piergiorgio Odifreddi and Karlheinz Stockhausen. Intervista a KARLHEINZ STOCKHAUSEN. http://www.piergiorgioodifreddi.it/wp-content/uploads/2010/10/stockhausen.pdf, 2012.

[24] Joonas Kokkonen. *String Quartet No. 2*. Espoo: Fazer Music, 1966.

[25] Edward Jurkowski. Joonas Kokkonen's "Free" Dodecaphonic Composition? A Study of the Passacaglia from the String Quartet No. 2. *STM-Online (Svensk Tidskrift för Musikforskning)*, 3, 2000.

[26] Richard Bunger. *Pianography: Fantasy on a Theme by Fibonacci*. Los Angeles, 1971.

[27] Per Nørgård. *Canon for Organ*. Copenhagen: Edition Wilhelm Hansen, 1974.

[28] Donald Walker. *Sixth Piano Sonata*. Walnut Creek, 1973.

[29] Matteo Meneghini. *Stria*, by John Chowning: Analysis of the Compositional Process. In *Proceedings of the XIV Colloquium on Musical Informatics*, Florence, May 2003.

[30] Curtis Roads and John Chowning. *Composers and the Computer*, chapter John Chowning On Composition, pages 16–25. Los Altos, William Kaufman, 1985.

[31] Brian Ferneyhough. Program notes from *Concert XX*. CONCERTEN Tot en Met, http://concerten.free.fr/m/c/program/XX.html, 2001.

[32] Brian Ferneyhough. *Time and Motion Study I*. New York: Edition Peters, 1977.

[33] Mikko Heiniö. *Duo per violino e pianoforte*. Helsinki: Edition Fazer, 1982.

[34] Clarence Barlow. *… until … version eight*. Cologne: Feedback Studio Verlag, 1981.

[35] Clarence Barlow. "Tschelpowisch." Score. 1996.

[36] Michael Gielen. *Streichquartett*. Frankfurt: Henry Litolff's Verlag / C. F. Peters, 1987.

[37] Sophia Gubaidulina. *Stimmen … Verstummen …*. London: Boosey & Hawkes, 1986.

[38] Vera Lukomsky and Sophia Gubaidulina. 'Hearing the Subconscious': Interview with Sofia Gubaidulina. *Tempo, New Series*, 209:27–31, July 1999.

[39] Haukur Tómasson. *Octette*. Reykjavik: Íslensk tónverkamiðstöð, 1987.

[40] Harold Blumenfeld. *Diluvial* from *Illuminations*. Saint Louois: MMB Music, 1992.

[41] William Hill. *Visions in Abstract Relativity*. Denver, 1999.

[42] Ramón García i Soler. *Scherzo con Fibonacci*. Albaida: Omnes Bands, 2005.

[43] Joseph F. Tyldesley. *Benedictus Deus*. Waltham, 2005.

[44] Kiyomitsu Odai. *Passacaglia di Fibonacci*. Santa Barbara, 2011.

[45] Da Jeong Choi. *Dream of a Thousand Keys: A Concerto for Piano and Orchestra*. Denton, 2011.