Fibonacci Pitch Sets: Beyond Mod 12

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0.0 Preliminaries

In my essay *Fibonacci Pitch Sets*, I attempted to create a thorough application of Fibonacci sequences to pitch structure using 12 tone equal temperament. Fibonacci pitch structures are not, however, limited to any single temperament; this essay continues the investigation of Fibonacci pitch sets.

1.0 Fibonacci Modulo 24

Considering the relation of prime sets to subsets discussed in my previous essay, it seems to me musically interesting to discuss the first superset of the mod 12 collection, the mod 24 collection. And considering the already widespread usage of such a temperament,[1] it seems a useful place to start. First we will look at the classic Fibonacci sequence. While in the mod 12 collection there exists one prime form of the Fibonacci sequence, there are two prime Fibonacci sequences in the mod 24 collection. These two unique sequences have a multiplicative relationship of 5. Let us take a look at f1 and f5 (mod 24):

f1	0	1	1	2	3	5	8	13	21	10	7	17	0	17	17	10	3	13	16	5	21	2	23	1
	0	23	23	22	21	19	16	11	3	14	17	7	0	7	7	14	21	11	8	19	3	22	1	23

f5	0	5	5	10	15	1	16	17	9	2	11	13	0	13	13	2	15	17	8	1	9	10	19	5
	0	19	19	14	9	23	8	7	15	22	13	11	0	11	11	22	9	7	16	23	15	14	5	19

The period of f1 and f5 is 24, the restricted period is 12, and the multiplier is 17. It was noted before that any multiple of a defective modulus is defective; 12 is a defective modulus, so 24 is as well. Table 1 shows the mod 24 collection, table 2 shows the inverse Fibonacci mod 24 collection. The mod 24 collection contains $24^2 = 576$ ordered dyads.

DEFINITION. *Let n = some integer. Let f_n denote a Fibonacci sequence of integers (not under a modulus) where the defining relation is $f_n + f_{n+1} = f_{n+2}$ for the successive terms of the sequence: $f_0, f_1, f_2, \ldots, f_n, \ldots$, and let $f_0 = 0$ and $f_1 = 1$. Let l_n denote the Lucas sequence, having the same defining relation as the Fibonacci sequence $l_n + l_{n+1} = l_{n+2}$, but where the starting values are $l_0 = 2$ and $l_1 = 1$. Let m = some integer used as a modulus.*

1.1 Some Properties of the Mod 24 Collection

In the mod 24 collection, two new important Fibonacci sequences come to our attention. The sequence {3, 1, 4, 5, 9, 14, 23, 37, 60, 97, 157, 254, …}, which we will call e_n,[2] has the relation $f_n + l_{n+1} = e_{n+2}$. And the sequence {5, 2, 7, 9, 16, 25, 41, 66, 107, 173, 280}, we will call m_n, has the relation $l_n + e_n = m_n$. Some more relations between these sequences are found in the notes.[3]

In the mod 12 collection, it was noticed that because of the defective modulus, f1 is missing residue 6, and l1 is missing residues 0 and 9. The impact of the defective modulus is heightened in mod 24. Table 3 shows the number of occurrences of each residue class in prime sets. f1 generates the unordered set of pitches {0, 1, 2, 3, 5, 7, 8, 10, 13, 16, 17, 21, 23}. Multiply this by five and we get the unordered pitch class set of f5, {0, 1, 2, 5, 8, 9, 10, 11, 13, 15, 16, 17, 19}. These pitch class sets are of cardinality 13. In l1 the unordered set is {1, 2, 3, 4, 5, 7, 10, 11, 13, 17, 18, 19, 20, 23}, multiplied by five we get {1, 2, 4, 5, 7, 10, 11, 13, 15, 17, 18, 19, 20, 23}. These sets are of cardinality 14. e1 and e5 produce the unordered sets {1, 3, 4, 5, 7, 9, 12, 13, 14, 17, 20, 22, 23} and {1, 4, 5, 11, 12, 13, 14, 15, 17, 19, 20, 21, 22} respectively. These sets are of cardinality 13. m1 and m5 have a fascinating property: their unordered set maps onto itself when multiplied by 5; {1, 2, 5, 7, 8, 9, 10, 11, 13, 16, 17, 18, 19, 21, 23}. In order for a set to map onto itself when multiplied by 5; it is required that for every member of the set *y*, there is another corresponding element in that set 5*y*, which occurs an equal number of times. m1 and m5, being of cardinality 15, have the greatest cardinality of any set in the mod 24 collection. This brings up an important question: what determines the number of residues for a given set? In answering this question, we begin to establish more basic properties of Fibonacci sequences and congruences modulo *m*.

The number of unique residues present in a general Fibonacci pitch class set represents the quantity of numbers smaller than the modulus m that are residues mod m of any number in that Fibonacci sequence. Thus if a residue t is present in a Fibonacci sequence g_n (mod) m, there must be some element $x \equiv t$ (mod m) in that Fibonacci sequence g_n. And because g_n is infinite, $x \equiv t$ (mod m) occurs an infinite amount of times. When a residue t occurs n times in a period p, there must be n unique numbers conforming to $x \equiv t$ (mod m) for every p elements in the sequence g_n.

2.0 Interval Cycles

Understanding the interval cycles is useful for understanding Fibonacci pitch sets; they have many similarities to Fibonacci pitch sets.[4] Table 4 shows the interval cycles for mod 24. Interval cycles are ordered sets, so inversionally related cycles can be distinguished; $C13_0$ is $C11_0$ inverted. Inversion in the case of interval cycles means the same thing as retrograde.[5] The number of prime cycles in a collection of interval cycles modulo m can be determined by the amount of numbers relatively prime to m that are less than or equal to m. Those familiar with number theory may recognize this as the Euler function $ø(m)$.[6] The Euler function $ø(m)$ denotes the number of integers less than or equal to the modulus m that are relatively prime to the modulus; for $m = 24$, $ø(m) = 8$. For every prime modulus p; $ø(p) = p - 1$. The Euler function denotes, for an equal temperament of m tones, the number of intervals (and multiplications) capable of generating complete systems of residues for modulo m.[7] This collection of $ø(m)$ residues forms a reduced residue system mod m.[8] In 24 tone temperament a reduced residue system is {1, 5, 7, 11, 13, 17, 19, 23}. The number of subcycles in an interval cycle collection is determined by $m - ø(m)$. In this collection there are 8 prime cycles, and 16 subcycles. It can be seen that the 24 tone scale has 8 equally tempered subsets; 24, 12, 8, 6, 4, 3, 2, and 1. The number of equally tempered subsets for modulo m is the number of positive divisors of m.

3.0 Prime Sets and Subsets

The prime sets in a mod m collection occupy a very special place in a collection; these are the sets that belong only to mod m. In such sets, every dyad is relatively prime to m.[9] There are 17 unique sets in this collection; 8 of these are prime sets. For each prime set there is another set related through

multiplication by 5; such sets will be called congruent prime sets. There are thus 4 pairs of congruent prime sets generated from the sequences $\{f_n, l_n, e_n, m_n\}$. For all prime sets the period is 24,[10] the multiplier is 17, and the restricted period is 12. One can discern the relations in all prime sets between multiplications by {1, 5, 7, 11, 13, 17, 19, 23} from the relative location of the dyads {0, 1}, {0, 5}, {0, 7}, {0, 11}, {0, 13}, {0, 17}, {0, 19}, and {0, 23} in f1 and f5. All the prime sets in tables 8 and 9 have been arranged so that multiplications by 5 are horizontally aligned.

The first equally tempered subset of the 24 tone scale is the 12 tone scale; every set in the mod 12 collection is present in the mod 24 collection.[11] In the mod 12 collection the multiplier was 5 for all sets; in fact, all subsets in the mod 24 collection have a multiplier of 5. In all the even sets (sets whose members are part of mod 12 collection), multiplying by 17 is equivalent to multiplying by 5. This is because $17 \equiv 5 \pmod{12}$. While it can not be said that the multiplier for the entire mod 24 collection is 17, multiplying by 17 always maps a sequence onto some part of itself.

The relations of subsets and supersets is considerably more complex in mod 24 than in mod 12. {f1, f5, e1, e5} map onto the same subsets when multiplied by the same number. {l1, l5, m1, m5} map onto the same subset when multiplied by the same number, but do not map onto the same subsets as {f1, f5, e1, e5} when multiplied by 2 or 3. The relationships between prime sets and subsets are shown graphically in table 5. In the mod 24 collection, the Lucas sequence has several subsets that are totally unrelated to the Fibonacci sequence; one might recall in the mod 12 collection that l1 is the only set in the collection not related to f1 by multiplication. The same is true of f1 and l1 in the mod 8 subcollection.

3.1 Another Significant Subcollection

The mod 8 collection is a significant enough subcollection of mod 24 to be of interest in itself; it comprises $8^2 = 64$ ordered dyads. The mod 8 prime sets are the only subsets in mod 24 not shared with the mod 12 collection. Let us first orient ourselves to mod 8 using table 6, which shows us the interval cycles. Here $\phi(8) = 4$, and the reduced residue system is {1, 3, 5, 7}. In the mod 24 collection, f1 (mod 24) and l1 (mod 24) remain unique under multiplication by 3; now let us look at the prime form of the Fibonacci sequence, and the Lucas sequence mod 8:

	0	1	1	2	3	5	0	5	5	2	7	1		2	1	3	4	7	3	2	5	7	4	3	7
f1													l1												
	0	7	7	6	5	3	0	3	3	6	1	7		6	7	5	4	1	5	6	3	1	4	5	1

Under mod 8 the period is 12, restricted period 6, and the multiplier is 5. It was noted before that all subsets of the mod 24 collection have a multiplier of 5. 8 is a defective modulus; residues 4 and 6 are not in f1, nor are residues 0 or 6 in l1. The mod 8 collection is shown in table 7. Because of their common divisors, every subset in the mod 8 collection is also in the mod 12 collection.

4.0 Invariances in the 24 Tone Collection

The defining relation $f_n + f_{n+1} = f_{n+2}$ and the commutative property of addition assure that each ordered dyad will be contained in an invariant trichord; for example {20, 4, 0} and {4, 20, 0}. Because $f_{n+2} - f_{n+1} = f_n$, the term preceding each ordered dyad (or invariant trichord) is the inversion of its invariant counterpart's preceding term; for example {1, 3, 4, 7} and {23, 4, 3, 7}. Table 8 graphs relations between invariant trichords in the prime sets. In this table, only the first 12 dyads need to be graphed; this is because their multiplication by 17 graphs them onto the same part of the invariant counterpart's restricted subset. The location of invariant trichords in f5 (or l5, e5, or m5) can be discerned in the same way; their locations correspond exactly to their congruent counterpart's location, because they are horizontally aligned in direct multiplicative correspondence with their congruent counterpart. The dotted line in the first example in the table shows this congruence.

Trichords beginning with identity dyads {*x, x, y*} have no invariant counterpart, with one exception; the trichord {12, 12, 0} overlaps with its two other invariant counterparts, and has 3 possible Fibonacci orderings. There are therefore 23 of these trichords. Any invariant trichord involving identities such as {*x*, 0, *x*} and {0, *x*, *x*} consequently intersects with its invariant counterpart; there are 23 pairs of such trichords in the collection. In the prime sets there are 190 pairs of invariant trichords that do not overlap in the sequence; they take the form {*x, y, z*} and {*y, x, z*}. The number of nonintersecting invariant trichords in the collection is 263.[12]

Table 9 shows occurrences of invariant tetrachords in prime sets. Any modulus that can be divided by 2 or 3 consequently contains invariant tetrachords. So, in mod 24, any Fibonacci tetrachord

whose second element is 8 (or 16) has 2 invariant counterparts. In mod 24 dividing the octave into 2 and 3 parts generates interval cycles $C12_x$ and $C8_x$. Each case of an unordered tetrachord occurring in three different places involves a subset $C8_x$. The second element is always either 8 (or 16); for example $C8_3$ is a subset of {3, 8, 11, 19}, {11, 8, 19, 3} and {19, 8, 3, 11}. There are 8 such tetrachords occurring in 3 places. We noted before that sets e1, e5, l1, and l5 do not contain residue 8. These sets do not exhibit this type of invariance. Any tetrachord beginning 12 has 1 invariant counterpart; for instance {12, 17, 5, 22} and {12, 5, 17, 22}. The only prime sets that contain residue 12 are e1, and e5; so these sets exhibit all invariances of this type. l1 and l5, which contain neither residues 8 or 12, consequently have no invariant tetrachords.

5.0 Fibonacci Pitch Sets in 24 Tones

With 24 tones in an octave, we can extend nonmodular Fibonacci sets slightly further into the sequence and the golden ratio can be more accurately expressed. These sets are indexed by the first two pitch integers *n* and *m* that are used, as they define the pitches that succeed. The symbol $F_{n, m}t$ represents nonmodular Fibonacci pitch sets; *t* denotes the cardinality of the set. Inverted (descending) sets are denoted by $F_{-n, -m}t$; so the set $F_{-1, -2}8$ = {-1, -2, -3, -5, -8, -13, -21, -34}. In normal form *n* and *m* are less than 24. $-F_{n, m}t$ represents inverse Fibonacci pitch sets. Table 10 shows an example of Fibonacci pitch sets based on l1 (mod 24). Table 11 shows the inverse pitch sets based on –f1 (mod 24). The range for these charts was set at a delightful 224 pitches; twice that of the mod 12 tables in the previous essay. With 24 pitches in the octave, the set $F_{0, 1}13$ {0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144} can express the golden ratio at 89/144 = .618055556…

6.0 Some Words on Fibonacci Mod 24

24 tone equal temperament has a very special place in our ears, being the first superset of the 12 tone scale. And the results achieved with Fibonacci mod 24 are excellent. There is clearly a new level of intricacy and expression achieved in this collection, and the Fibonacci mod 24 collection sounds unmistakably Fibonacci. This type of resemblance in sound is a different type from that achieved through simply dividing a 12 tone pitch set by 2. In the Fibonacci collection we have not divided the mod 12 sets

by 2; doing this actually results in a totally different pitch class set with quasi Fibonacci properties.[13] This resemblance in sound can therefore only be attributed to the Fibonacci defining relation $f_n + f_{n+1} \equiv f_{n+2}$ (mod) m.

7.0 An Overview of Fibonacci Modulo 19

Another equal temperament receiving attention recently is 19 tone.[14] I thought it of interest because of its sharp contrast to 24 tone temperament. This temperament will not be discussed in the same level of detail as the previous temperaments; but I will give a brief overview to demonstrate the effect a prime modulus has on Fibonacci pitch sets.[15] Understanding this temperament will give the reader a good idea of the range of possibilities Fibonacci pitch sets encompass. Fibonacci mod 19 will conclude this essay.

Table 12 shows interval cycles for mod 19. Here ø(19) = 18, so every cycle excluding the identity cycle $C0_0$ is prime, and every Fibonacci set in the mod 19 collection is prime. f1 (mod 19) is shown below; the entire collection is shown in table 13.

	0	1	1	2	3	5	8	13	2	15	17	13	11	5	16	2	18	1
f1																		
	0	18	18	17	16	14	8	6	17	4	2	6	8	14	3	17	1	18

The collection contains $19^2 = 361$ ordered dyads; multiplications of f1 make up the vast majority of the collection, accounting for $18^2 = 324$ of the dyads. Notice the Fibonacci sequence contains the Lucas sequence exactly at its midpoint when multiplied by 9 (look in the middle of {f9: 2, 1, 3, 4, 7, 11, …}); so the midpoint of all Fibonacci sequences mod 19 is the Lucas sequence. All multiplications of f1, excluding {f1 * 0}, have a period of 18. No multiplications of f1 have a restricted period. 19 is a defective modulus; all prime moduli are defective except 2, 3, 5, and 7.[16] Because mod 19 can not be be divided by 3 or 2 (moduli divisible by 3 or 2 have been shown to create invariant tetrachords in any temperament where this is possible, see Fibonacci Pitch Sets 2.6), there are no invariant tetrachords in the mod 19 collection.

The most curious part of the collection is the remaining 2 sets named g1 and h1. In mod 19 there are 36 ordered dyads not related to f1 through multiplication. These sets seem to behave radically differently from any sets I have discussed before. g1 has a period of 9, showing that under certain moduli,

some general Fibonacci sequences have a period that is an odd number. h1 contains its own inversion. Both sets contain every residue except 0, meaning that in the collection of multiplications of f1, every residue occurs an equal amount of times. In the collection of prime sets as a whole, residue 0 occurs one less time than every other residue. Both of these phenomena are true for all prime moduli.[17] These sets have a restricted period of 1; g1 has multiplier 5 and h1 has multiplier 15. They map onto themselves when multiplied by any number besides 0, so they contain 18 similar (congruent) sets inside themselves.

8.0 Concluding Remarks

One can see the drastic effect a prime temperament has on Fibonacci pitch sets, and on chromatic music in general for that matter. Although the sound of the mod 19 collection is still very "Fibonacci", there are no subsets; everything is prime, and the vast majority of the collection is made up of multiplications of f1. The collection as a whole sounds more similar for this reason. This could be seen as a musical advantage, or might be seen as a disadvantage. The mod 24 collection, in contrast, has an incredible sonic variety within its scope, due to its rich variety of subsets.

This essay suggests questions of what lies beyond these bounds. As we slice the octave into smaller pieces, nonmodular Fibonacci pitch sets get closer and closer to the golden pitch set. An irrational pitch set in which each successive interval is exactly $(1 + \sqrt{5})/ 2$ of the last.[17] My next essay will continue the investigation of Fibonacci pitch sets and will make a detailed investigation of golden pitch sets.

Notes

1. The fascination of musicians with quarter tones dates as far back as the baroque era. Giuseppe Tartini (1692 – 1770) proposed symbols commonly used today to indicate ¼ sharps and flats. An interesting writing on the history of microtonality is found in the preface to Gardner Read's *20th Century Microtonal Notation.* A selected list of the composers who have used this temperament in their works includes; Charles Ives, Krzysztof Penderecki, Witold Lutoslawski, Brian Ferneyhough, Iannis Xenakis, Cristobal Halffter, and Alois Hába.
2. e_n was chosen to represent the sequence {3, 1, 4, 5, 9, 14, 23, 37, 60, 97, 157, 254, …} because, in the mod 24 collection, this sequence also contains the Evangelists' sequence {3, 2, 5, 7, 12, 19, 31, 50, 81, 131, 212 …} when multiplied by 5; but remember, here $e_0 = 3$, and $e_1 = 1$.
3. These four sequences are the "seeds" of the mod 24 pitch class sets. More relations include $f_n + f_{n+5} = e_{n+3}$, $l_n + f_{n+5} = m_{n+2}$, and $e_n + m_{n+1} = f_{n+5}$.
4. Both the interval cycles and Fibonacci pitch sets are arithmetic, modular residue cycles. I find it interesting that the collection of interval cycles, like the Fibonacci collection, contains every ordered dyad.
5. Consider this fact in relation to Fibonacci sequences, where retrograde means the inversion of the defining relation.
6. Leonhard Euler (1707 – 1783) was a Swiss mathematician, known for his important contributions in the fields of geometry, calculus, and number theory.
7. A complete system of residues modulo *m* is a set that contains every possible residue class of *m*; for instance {2, 3, 4, 5, 6, 7, 8, 9} is a complete system of residues mod 8. The complete collection of Fibonacci pitch sets mod *m* is also called a complete Fibonacci system mod *m*.
8. In general, a reduced residue system modulo *m* is created by deleting from a complete residue system those elements that are relatively prime to the modulus *m*; {3, 5, 7, 9} is a reduced residue system mod 8. Similarly, the collection of prime sets in a Fibonacci mod *m* collection can be called a reduced Fibonacci system mod *m*.
9. Two numbers are said to be relatively prime if their greatest common denominator is 1.

10. The longest period any general Fibonacci sequence can have under a modulus *m* is that of the classic Fibonacci sequence mod *m*. An intriguing proof of this can be found in *Fibonacci and Lucas Numbers and the Golden Section*, by Steven Vadja.
11. In the mod 24 collection these sequences will appear to be multiplied by 2.
12. In the mod 12 collection there were 55 pairs of invariant trichords that did not intersect. Since the mod 8 collection is the only subset mod 24 does not share with 12, the 18 remaining invariant trichords are found in the mod 8 prime sets. 190 + 55 + 18 = 263.
13. The resulting pitch class set is not of interest to the author, the reader is encouraged not to investigate the resulting pitch class set of dividing f1 mod 12 by 2.
14. Among the many reasons this temperament attracts attention, many are interested in this temperament because it can function in diatonic tonality and many of its intervals are close to "pure". I am interested primarily because it is a prime temperament. In fact, while diatonically 19 tone behaves extremely similar to 12 tone, here we have shown that chromatically (and mathematically) it behaves radically different. Some composers having used 19 tone temperament include Easley Blackwood, Alois Hába, Joseph Yasser, and Ivan Wyschnegradsky.
15. Prime moduli have been the subject of a great deal of research regarding Fibonacci sequences. The most important research papers in this area are Wall's *Fibonacci Series Modulo m*, and Bruckner's *Fibonacci Sequence Modulo A Prime p ≡ 3 (mod 4)*.
16. Bruckner's Theorem 1970.
17. The additive identity guarantees that this phenomenon is not true for non prime moduli. 0 can not occur ($m - 1$) times in the prime portion of a collection when the modulus is not prime; non prime moduli gaurantee that 0 has occurred in a subset besides f0, furthermore, 0 has occurred $m - \text{ø}(m)$ times in subsets including f0.
18. $(1 + \sqrt{5})/ 2 = 1.6180339887498948482045868343 66\ldots$

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