Base Phi Representations:
Base Phi Minimum Binary, Base Phi Maximum Binary

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the sequences: There exists a binary non-standard positional number system which uses the golden ratio as a base. In this system, powers of the golden ratio are used instead of the powers of ten used in the decimal system. Despite the base being irrational, every integer has a unique finite representation.

The golden ratio values are symbolized by Phi = 1.618033989…, and phi = .618033989…. The following diagram shows the values of the powers of Phi, and illustrates an interesting relation to the Fibonacci sequence:

Phi^5	8 + 5phi	11.0901699…
Phi^4	5 + 3phi	6.85410197…
Phi^3	3 + 2phi	4.23606798…
Phi^2	2 + 1phi	2.61803399…
Phi^1	1 + 1phi	1.61803399…
Phi^0	1 + 0phi	1.00000000…
Phi^{-1}	0 + 1phi	0.61803399…
Phi^{-2}	1 - 1phi	0.38196601…
Phi^{-3}	-1 + 2phi	0.23606798…
Phi^{-4}	2 - 3phi	0.14589803…
Phi^{-5}	-3 + 5phi	0.09016994…

This relation to the Fibonacci sequence can be captured in the equation:

$Phi^n = Fib_{n+1} + Fib_n*phi$

Here are the integers 1 through 11 in base Phi:

decimal	powers of Phi	base Phi
1	Phi^0	1
2	Phi^1+Phi^{-2}	10.01
3	Phi^2+Phi^{-2}	100.01
4	$Phi^2+Phi^0+Phi^{-2}$	101.01
5	$Phi^3+Phi^{-1}+Phi^{-4}$	1000.1001
6	$Phi^3+Phi^1+Phi^{-4}$	1010.0001
7	Phi^4+Phi^{-4}	10000.0001
8	$Phi^4+Phi^0+Phi^{-4}$	10001.0001
9	$Phi^4+Phi^1+Phi^{-2}+Phi^{-4}$	10010.0101
10	$Phi^4+Phi^2+Phi^{-2}+Phi^{-4}$	10100.0101
11	$Phi^4+Phi^2+Phi^0+Phi^{-2}+Phi^{-4}$	10101.0101

A radix point '.' is used to separate the powers of Phi into those greater than or equal to 1 and those less than 1.

As a rule, the base Phi representations of integers shown above never contain the digit sequence "11" (or "1.1"). Put in

other words, they never contain any consecutive powers of Phi (Phi^4 and Phi^3, for example). This system of representing integers and the piece based on it I will call *Base Phi Minimum Binary*.[1]

Because in this system 100 is equal to 011 (this follows from $Phi^n = Phi^{n-1} + Phi^{n-2}$), it is possible to use a replacing-algorithm to eliminate every occurrence of the digit sequence "00" so that every integer also has an alternate unique finite representation which contains no instance of "00". This system of representations and the piece based on it I will call *Base Phi Maximum Binary*.

Both sequences are shown below:

	Minimum	*Maximum*
0	0.	0.
1	1.	1.
2	10.01	1.11
3	100.01	11.01
4	101.01	101.01
5	1000.1001	101.1111
6	1010.0001	111.0111
7	10000.0001	1010.1101
8	10001.0001	1011.1101
9	10010.0101	1101.1101
10	10100.0101	1111.0101
11	10101.0101	10101.0101
12	100000.101001	10101.111111
13	100010.001001	10111.011111
14	100100.001001	11010.110111
15	100101.001001	11011.110111
16	101000.100001	11101.110111
17	101010.000001	11111.010111
18	1000000.000001	101010.101101
19	1000001.000001	101011.101101
20	1000010.010001	101101.101101
21	1000100.010001	101110.111101
22	1000101.010001	101111.111101
23	1001000.100101	110101.111101
24	1001010.000101	110111.011101
25	1010000.000101	111010.110101
26	1010001.000101	111011.110101
27	1010010.010101	111101.110101
28	1010100.010101	111111.010101
29	1010101.010101	1010101.010101

More terms (as many as 24,476) and other files for performing this work can be found at the following web addresses:[2]

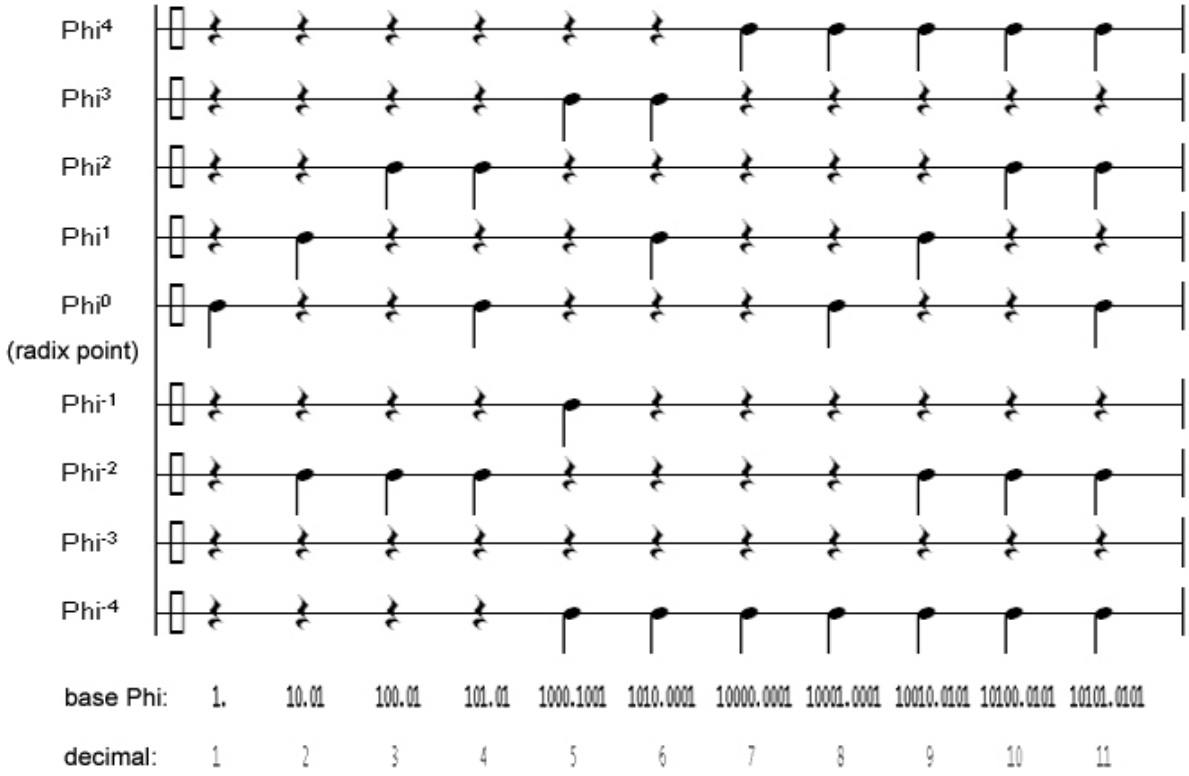
http://caseymongoven.com/catalogue/B416.html

and http://caseymongoven.com/catalogue/B417.html

the pieces: This score is for two works (catalogue B416 and B417), and these pieces can be performed separately or on the same concert. The order of performance is not important. Freedom is given to the performers to determine compositional

aspects of the works with certain restrictions to be explained.

In this work, each player is assigned a power of Phi. Each performer plays their note - the same note - every time their assigned power of Phi occurs in the integer being expressed. The following example of *Base Phi Minimum Binary* in traditional notation should help make this clear:



The above example is a short performance (only 11 integers) for 8 players.

There are 3 versions of this work: one for percussion instruments of indefinite pitch, one for percussion instruments of definite pitch, and one for pitched instruments (in which some percussion instruments are allowed as well). Each player plays the same type of instrument in every case. The three versions are explained separately below. First, the elements common in all versions will be explained.

number of integers played: Each member of the sequence is given equal duration. The duration of each member is up to the performers to decide, but should be at least 70 beats per minute (.857 seconds or faster). The players are to start with the representation for 1 and play through in increasing order (i.e. in decimal notation 1,2,3,4,5,6,7,…). A Lucas number of integers are to be played:

2,1,3,4,7,11,18,29,47,76,123,199,322,521,843,1364,2207,3571,…

One can imagine that very long performances are possible. If both works are played, the same number of integers should be played in both.

number of players: The number of integers to be played determines the number of players required. If for example, 11 integers are to be played in the minimum representation, this will require 8 players because 10101.0101 has 9 digits and the second to last digit is never used in the minimum representation (this can be seen in the example in traditional notation above; Phi^{-3} has only rests). Note, however, that for maximum representations a player is needed to represent every digit.

spacing on stage: In performance, regardless of the number of players needed, there should be a person standing in for the power of Phi not used in the minimum representations - with instrument if possible.

There are 3 options for spacing in this work. In each spacing, the players line up in a straight line from stage left to stage right. The bigger the space where playing and the more reverberation, the larger the space between the performers should be. Space is always measured from the center of the body. The positions should be marked on stage beforehand as in one of the following options:

1. Even spacing is used. There should be at least 1.4 meters, performer to performer. For example:

Φ^4----Φ^3----Φ^2----Φ^1----Φ^0----Φ^{-1}----Φ^{-2}----Φ^{-3}----Φ^{-4}

2. Even spacing with a radix point is used. There should be at least 1.4 meters between each performer and the space representing the radix point should be at least 2 times the distance between each performer and can be much greater. It is optional to use an object to represent the radix point; if used, this should be placed directly in between performers Φ^0 and Φ^{-1}. For example:

Φ^4----Φ^3----Φ^2----Φ^1----Φ^0----‘.’----Φ^{-1}----Φ^{-2}----Φ^{-3}----Φ^{-4}

3. Spacing based on the golden ratio is used. In this spacing, the distance between the players increases from right to left based on the golden ratio. The space between the two 2 right-most players cannot be less than 75 centimeters. If x represents the space between the players at the right-most end, then the distance for between the players is figured as such:

Phi^3*x Phi^2*x $Phi*x$ x

Φ^2----------Φ^1----------Φ^0-------Φ^{-1}----Φ^{-2}

In the 3rd spacing option, the smaller space must be between the smaller powers of Phi as shown. This option requires more space to perform than the others.

Any of these spacings given as examples may be inverted from left to right; the lowest value can be on the left or right side.

synchronization: Some electronic means of synchronization must be used in performance for the sake of large scale precision. This can be a conductor who is connected to a click track, a click track for the performers, or a blinking light. If a conductor is used, he should be placed in a position that does not block the sound field of the performers.

articulation, dynamic, manner of playing: The group should agree on a dynamic level which will stay the same for the duration of the work. Each player plays their note exactly the same each time: exactly as loud and in the exact same manner as the other players. The posture of the musicians should be as uniform as possible. The attack should be crisp. For instruments with decay times longer than the duration of one member of the sequence, the note should be dampened each time it is played. Each note should last an absolute maximum of 1 member of the sequence (i.e. in the example above in traditional notation, at maximum as long as a quarter note).

rehearsal, performance suggestions: After the performers have agreed on how to perform the work, preparations must be made by each player. Each player should practice their part without the group beforehand. Although it is up to the performers, I would recommend not using standard notation in performing this work. In faster versions it might be necessary for some performers to write out their part in some standard notation. But it is best to perform the work with a list of numbers in front of oneself, with the part to be performed in bold print:

1	1.	1.
2	10.01	1.11
3	**1**00.01	11.01
4	**1**01.01	**1**01.01
5	1**0**00.1001	**1**01.1111
6	1**0**10.0001	**1**11.0111
7	10**0**0.0001	1**0**10.1101
8	10**0**1.0001	1**0**11.1101
9	10**0**10.0101	1**1**01.1101
10	10**1**00.0101	1**1**11.0101
11	10**1**01.0101	10**1**01.0101
12	100**0**0.101001	10**1**01.111111
13	100**0**10.001001	10**1**11.011111
14	100**1**00.001001	11**0**10.110111
15	100**1**01.001001	11**0**11.110111
16	101**0**0.100001	11**1**01.110111
17	101**0**10.000001	11**1**11.010111
18	1000**0**0.000001	101**0**10.101101

The 0s are, of course rests, and 1s attacks. Such notation for this work is available at the web addresses given above with up to 24,476 members.[3]

The difficult task of performing long versions of this work can be made easier by recognizing the regularity and patterns in the digits. Numbers indicating the length of the runs of 1s

and Os might be hand-written into the margin of the score. The players at the ends must be prepared to play long runs of notes without losing count. It can be very helpful to practice this work at slower tempos. Also of great importance is to check one's counting by listening to the players directly on one's left or right side, because their parts are most visible in the score.

Players who find the lack of barlines (and standard notation) in the work confusing might draw in lines into the score as such:

1	1.	1.
2	10.01	1.11
3	100.01	11.01
4	101.01	101.01
5	1000.1001	101.1111
6	1010.0001	111.0111
7	10000.0001	1010.1101
8	10001.0001	1011.1101
9	10010.0101	1101.1101
10	10100.0101	1111.0101
11	10101.0101	10101.0101
12	100000.101001	10101.111111
13	100010.001001	10111.011111
14	100100.001001	11010.110111
15	100101.001001	11011.110111
16	101000.100001	11101.110111
17	101010.000001	11111.010111
18	1000000.000001	10101.101101

The score might also be turned on its side for a player who finds it confusing to read music vertically.

The conductor or director overseeing the group in rehearsal should keep a printed list of the members of the sequence as such at hand:

1	1.	1.
2	10.01	1.11
3	100.01	11.01
4	101.01	101.01
5	1000.1001	101.1111
6	1010.0001	111.0111
7	10000.0001	1010.1101
8	10001.0001	1011.1101
9	10010.0101	1101.1101
10	10100.0101	1111.0101
11	10101.0101	10101.0101
12	100000.101001	10101.111111
13	100010.001001	10111.011111
14	100100.001001	11010.110111
15	100101.001001	11011.110111
16	101000.100001	11101.110111
17	101010.000001	11111.010111
18	1000000.000001	10101.101101

The orientation on the page should be the same as the performers on stage (i.e. if the smallest power of Phi is

stage-left, then the traditional notation should be inverted on the page).

The director can help in that he periodically gives the beat number using a hand-signal in performance, or saying the number out loud in rehearsal. The director should make the performers aware of the most important structural points in the work. At the same time he should stress that the players should not play any differently at these points. The most important structural points in these works occur around Lucas numbers (2,1,3,4,7,11,18,…).

for percussion instruments of indefinite pitch

All instruments used must be of the same type and dimensions. The instruments must blend well together and sound very similar to one another; no instrument should stick out in the group. The instruments must be capable of producing a crisp attack with no perceivable pitch. Instruments such as maracas, ratchets and cymbals are not to be used. Instruments which are in the so-called gray area of pitch-definiteness, such as tom-toms, are acceptable if a technique can be found which suppresses the perceived pitch (a rimshot, for example, can be effective in covering the perceived pitch of a tom-tom). Performances on unconventional instruments are encouraged.

for percussion instruments of definite pitch

All instruments used must be of the same type and dimensions. The players must agree on a single pitch for the group to use; each performer plays the same pitch. This pitch should be at least 18.5 Hertz. The pitch may contain complex overtones as long as one and the same pitch is predominant from all instruments. The instruments must blend well together and sound very similar to one another; no instrument should stick out in the group. The instruments must be capable of producing a crisp attack. Examples of acceptable instruments include crystal glasses or glass bottles (struck with a mallet), handbells, steel drums, crotales, and timpani. Performances on unconventional instruments are encouraged.

for pitched instruments

All instruments used must be of the same type and dimensions. The instruments must be able to be tuned – and played - to a high degree of intonational precision (within 2 cents). Unacceptable instruments include all standard orchestral brass and woodwind instruments. The instrument must be able to produce a sharp attack. Most plucked and bowed instruments are acceptable. Percussion instruments such as crystal glasses may be used, timpani are not acceptable.

The group decides on a temperament and tuning – this should suit the instruments. The temperament must be based on the following equation, in which n is an integer 1 through 12:

$1 + \mathrm{phi}^{n}$

Here is an example for violins in the 10th temperament (1 + phi^{10}), and one for cellos in the 11th (right side). These examples use only open strings:

player	string IV tuned to
Phi^{6}	185.00 Hz
Phi^{5}	186.50 Hz
Phi^{4}	188.02 Hz
Phi^{3}	189.55 Hz
Phi^{2}	191.09 Hz
Phi^{1}	192.64 Hz
Phi^{0}	194.21 Hz
Phi^{-1}	195.79 Hz
Phi^{-2}	197.38 Hz
Phi^{-3}	198.99 Hz
Phi^{-4}	200.60 Hz
Phi^{-5}	202.24 Hz
Phi^{-6}	203.88 Hz

player	string I tuned to
Phi^{6}	207.65 Hz
Phi^{5}	208.69 Hz
Phi^{4}	209.74 Hz
Phi^{3}	210.80 Hz
Phi^{2}	211.86 Hz
Phi^{1}	212.92 Hz
Phi^{0}	213.99 Hz
Phi^{-1}	215.07 Hz
Phi^{-2}	216.15 Hz
Phi^{-3}	217.23 Hz
Phi^{-4}	218.32 Hz
Phi^{-5}	219.42 Hz
Phi^{-6}	220.52 Hz

The lowest to highest pitches shown in the diagrams above can be inverted as well, so that the smaller powers of Phi get the lower pitch.

This table shows the approximate value in cents of the temperaments which can be used. This table is only for convenience and should not be used to calculate the tuning; this should be done with a calculator using 1 + phi^{n}.

1 + phi^{1}	833 cents
1 + phi^{2}	560 cents
1 + phi^{3}	367 cents
1 + phi^{4}	236 cents
1 + phi^{5}	149 cents
1 + phi^{6}	94 cents
1 + phi^{7}	59 cents
1 + phi^{8}	36 cents
1 + phi^{9}	23 cents
1 + phi^{10}	14 cents
1 + phi^{11}	9 cents
1 + phi^{12}	5 cents

Only instruments which can be tuned to a very high degree of accuracy (guitars or cellos, for example) should use the temperaments with the smallest intervals. Instruments should be tuned offstage using the same articulation used in the performance.

If stringed instruments are used, the players should all play on the same string. Exceptions to this rule are the first 3 temperaments, which can use different strings; here the players should choose the strings carefully in a way to create the greatest uniformity in sound. The players should either 1) all use open strings or 2) use no open strings. The temperaments 10 through 12 can be used effectively tuning the same open string slightly apart on the different instruments,

as shown in the examples. This has the advantage of great intonational accuracy.

Dedication

This work is dedicated to English mathematician Ron Knott, who introduced me to these sequences.

Notes

1. The titles on programs should read as one of the following:

Base Phi Representations:
Base Phi Minimum Binary, Base Phi Maximum Binary

or Base Phi Representations:
Base Phi Maximum Binary, Base Phi Minimum Binary

or Base Phi Representations: Base Phi Minimum Binary

or Base Phi Representations: Base Phi Maximum Binary

2. More information on Fibonacci numbers, the golden ratio, and these sequences can be found on Ron Knott's page:

http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/phigits.html

3. To give one an idea of how long different versions of this work would be, here is a small table showing various possibilities. The values given are rounded to the nearest second:

	number of integers used												
bpm	***76***	***123***	***199***	***322***	***521***	***843***	***1364***	***2207***	***3571***	***5778***	***9349***	***15127***	***24476***
70	65 sec.	105	171	276	447	723	1169	1892	3061	4953	8013	12966	20979
90	51	82	133	215	347	562	909	1471	2381	3852	6233	10085	16317
110	41	67	109	176	284	460	744	1204	1948	3152	5099	8251	13351
130	35	57	92	149	240	389	630	1019	1648	2667	4315	6982	11297
150	30	49	80	129	208	337	546	883	1428	2311	3740	6051	9790
170	27	43	70	114	184	298	481	779	1260	2039	3300	5339	8639
190	24	39	63	102	165	266	431	697	1128	1825	2952	4777	7729
210	22	35	57	92	149	241	390	631	1020	1651	2671	4322	6993
230	20	32	52	84	136	220	356	576	932	1507	2439	3946	6385
250	18	30	48	77	125	202	327	530	857	1387	2244	3630	5874
270	17	27	44	72	116	187	303	490	794	1284	2078	3362	5439
290	16	25	41	67	108	174	282	457	739	1195	1934	3130	5064
310	15	24	39	62	101	163	264	427	691	1118	1809	2928	4737
330	14	22	36	59	95	153	248	401	649	1051	1700	2750	4450
350	13	21	34	55	89	145	234	378	612	991	1603	2593	4196
370	12	20	32	52	84	137	221	358	579	937	1516	2453	3969
390	12	19	31	50	80	130	210	340	549	889	1438	2327	3766
410	11	18	29	47	76	123	200	323	523	846	1368	2214	3582
430	11	17	28	45	73	118	190	308	498	806	1305	2111	3415
450	10	16	27	43	69	112	182	294	476	770	1247	2017	3263
470	10	16	25	41	67	108	174	282	456	738	1193	1931	3125
490	9	15	24	39	64	103	167	270	437	708	1145	1852	2997